

DESY 10-220
 DO-TH 11/13
 SFB/CPP-11/42
 LPN 11/41
 July 2011

Two-loop QED Operator Matrix Elements with Massive External Fermion Lines

Johannes Blümlein^a, Abilio De Freitas^{a,b,1}, and Wilhelmus van Neerven^{c, 2}

^a*DESY, Zeuthen, Platanenallee 6, D-15735 Zeuthen, Germany*

^b*Departamento de Física, Universidad Simón Bolívar,
 Apartado Postal 89000, Caracas 1080-A, Venezuela.*

^c*Instituut-Lorentz, Universiteit Leiden,
 P.O. Box 9506, 2300 HA Leiden, The Netherlands*

July 26, 2011

Abstract

The two-loop massive operator matrix elements for the fermionic local twist-2 operators with external massive fermion lines in Quantum Electrodynamics (QED) are calculated up to the constant terms in the dimensional parameter $\varepsilon = D - 4$. We investigate the hypothesis of Ref. [1] that the 2-loop QED initial state corrections to e^+e^- annihilation into a virtual neutral gauge boson, except power corrections of $O((m_f^2/s)^k)$, $k \geq 1$, can be represented in terms of these matrix elements and the massless 2-loop Wilson coefficients of the Drell-Yan process.

¹Alexander-von-Humboldt Fellow.

²deceased.

1 Introduction

The QED corrections for differential distributions in e^+e^- annihilation and other high energy reactions in which electrons or positrons participate, are particularly large due to the presence of physical logarithms $\ln(M^2/m_e^2)$, with M a characteristic scale of the process and m_e the electron mass.³ Therefore, it is necessary to account for the QED initial state corrections up to $O(\alpha^2)$ for precision measurements in the various energy regimes in e^+e^- annihilation having been explored so far and those which are planned to be investigated in the future, cf. [3–12]. For the corrections to the inclusive Born cross section $\sigma(s)$, with s the center of mass (cms) energy squared, power corrections $\propto (m_e^2/s)^k$, $k \geq 1$ can be safely disregarded. While the $O(\alpha)$ corrections are known for a large number of reactions, cf. [13], the corrections beyond the universal contributions $O((\alpha L)^k)$, $1 \leq k \leq 5$ [14, 15], to higher orders, were only calculated analytically once at 2-loop order in Ref. [1].⁴ Besides the logarithmic orders $O(\alpha^2 L^2, \alpha^2 L)$ with $L = \ln(s/m_e^2)$, the constant terms $O(\alpha^2)$ are of interest.

In Ref. [1] it has been proposed that the 2-loop corrections can be calculated using a factorization-representation of the scattering cross section in terms of Mellin convolutions of massive local operator matrix elements (OMEs) and the corresponding massless Wilson coefficients, which are those of the massless Drell–Yan process [20, 21]. The operator matrix elements are formed by the local twist-2 fermionic operators, being obtained in the light cone formalism [22], between massive on-shell electron states. These operator matrix elements bear all the mass dependence, μ^2/m_e^2 , and are universal quantities. The massless Wilson coefficients account for the process dependence and are functions of the ratio \hat{s}/μ^2 , with \hat{s} the sub-system cms energy squared. Here, μ^2 denotes a factorization scale.

In case of the heavy flavor corrections to deep-inelastic scattering the above method was used to calculate the massive Wilson coefficients in the region $Q^2 \gg m_Q^2$, with Q^2 the virtuality of the exchanged gauge boson and m_Q the heavy quark mass. It has been shown that this description yields all but the power suppressed contributions in Refs. [18, 19] comparing to the complete semi-analytic calculation [23] at $O(\alpha_s^2)$.⁵ In this case the massive OMEs are formed between *massless* on-shell quark and gluon states. In Refs. [26–28] 2- and 3-loop heavy flavor corrections for different unpolarized and polarized nucleon structure functions and transversity, respectively for their moments, have been calculated.

In the present paper we compute the $O(\alpha^2)$ local OMEs with massive external fermions in QED up to the constant part. As a by-product we obtain the QED contributions to the 2-loop non-singlet and pure-singlet anomalous dimensions, within a massive calculation. We investigate, to which extent the decomposition, having been proposed in Ref. [1], in terms of massive local OMEs and massless Wilson coefficients is possible.

The paper is organized as follows. In Section 2 we summarize the decomposition of the initial state corrections to the inclusive e^+e^- annihilation cross section into neutral vector bosons to $O(\alpha^2)$ using the renormalization group method, cf. [1]. The renormalization of the OMEs is described in Section 3. In Section 4 we compute the OMEs to $O(\alpha)$. The details of the calculation of the $O(\alpha^2)$ corrections to the OMEs are given in Section 5. We discuss the structure of the contributions to the differential scattering cross section $d\sigma_{e^+e^-}/d\hat{s}$ and compare to the results in the literature. Section 6 contains the conclusions. Technical aspects are summarized in the

³ This also applies to the QED corrections in eN scattering, cf. [2].

⁴ There are only a few other complete analytic $O(\alpha^2)$ two-loop calculations with massive particles available. Examples are the heavy quark fragmentation functions [16], the μ -lepton decay spectrum [17] to $O(\alpha^2 L)$, while the $O(\alpha^2)$ contribution was given numerically, and the heavy flavor Wilson coefficients for large virtualities [18, 19].

⁵ For a fast and precise numerical implementations of these corrections in Mellin space cf. [24].

appendices.

2 The Renormalization Group Method

The $O(\alpha^2)$ QED initial state corrections to the e^+e^- annihilation cross section into virtual neutral gauge bosons (γ^*, Z^*) in the limit $s \gg m_e^2$ can be expressed in the following form [1] :

$$\frac{d\sigma_{e^+e^-}}{ds'} = \frac{d\sigma_{e^+e^-}^{\text{I}}}{ds'} + \frac{d\sigma_{e^+e^-}^{\text{II}}}{ds'} + \frac{d\sigma_{e^+e^-}^{\text{III}}}{ds'} . \quad (1)$$

Here, s' denotes the invariant mass of the virtual vector boson and s the cms energy of the process,

$$s' = xs, \quad x \in [0, 1] . \quad (2)$$

The term I refers to the photon radiation contributions, II to the flavor non-singlet contribution due to fermion pair production, and III to the corresponding flavor pure-singlet contribution.⁶ One may represent the scattering cross section in Mellin space by applying the integral transform

$$\widehat{f}(N) = \int_0^1 dx x^{N-1} f\left(x = \frac{s'}{s}\right) , \quad (3)$$

with

$$\frac{\widehat{d\sigma}}{ds'}(N) = \int_0^1 dx x^{N-1} \frac{d\sigma}{ds'}(xs) , \quad (4)$$

$$\widehat{\sigma}_0(N) = \int_0^1 dx x^{N-1} \sigma_0(xs) . \quad (5)$$

Since in the present calculation power corrections of $O(m_e^2/s)$ are disregarded, the following principle structure [1] with respect to the scales m_e^2, s and a factorization scale μ^2 is obtained, [1] :

$$\begin{aligned} \frac{\widehat{d\sigma}}{ds'}(N) = \frac{1}{s} \widehat{\sigma}_0(N) \times & \left[\Gamma_{e^+e^+} \left(N, \frac{\mu^2}{m_e^2} \right) \tilde{\sigma}_{e^+e^-} \left(N, \frac{s'}{\mu^2} \right) \Gamma_{e^-e^-} \left(N, \frac{\mu^2}{m_e^2} \right) \right. \\ & + \Gamma_{\gamma e^+} \left(N, \frac{\mu^2}{m_e^2} \right) \tilde{\sigma}_{e^-\gamma} \left(N, \frac{s'}{\mu^2} \right) \Gamma_{e^-e^-} \left(N, \frac{\mu^2}{m_e^2} \right) \\ & + \Gamma_{e^+e^+} \left(N, \frac{\mu^2}{m_e^2} \right) \tilde{\sigma}_{e^+\gamma} \left(N, \frac{s'}{\mu^2} \right) \Gamma_{e^-\gamma} \left(N, \frac{\mu^2}{m_e^2} \right) \\ & \left. + \Gamma_{e^+\gamma} \left(N, \frac{\mu^2}{m_e^2} \right) \tilde{\sigma}_{\gamma\gamma} \left(N, \frac{s'}{\mu^2} \right) \Gamma_{e^-\gamma} \left(N, \frac{\mu^2}{m_e^2} \right) \right] . \quad (6) \end{aligned}$$

To $O(\alpha^2)$ the last process in (6) does not contribute. Here $\sigma_0(s)$ denotes the Born cross section for e^+e^- annihilation into a virtual gauge boson (γ, Z) which decays into a fermion pair $f\bar{f}$, see e.g. [29],

$$\frac{d\sigma^{(0)}(s)}{d\Omega} = \frac{\alpha^2}{4s} N_{C,f} \sqrt{1 - \frac{4m_f^2}{s}} \times$$

⁶In Ref. [1] four contributions were considered dividing those to process I into two pieces according to the genuine $2 \rightarrow 3$ particle scattering cross sections.

$$\times \left[\left(1 + \cos^2 \theta + \frac{4m_f^2}{s} \sin^2 \theta \right) G_1(s) - \frac{8m_f^2}{s} G_2(s) + 2\sqrt{1 - \frac{4m_f^2}{s}} \cos \theta G_3(s) \right] \quad (7)$$

$$\sigma^{(0)}(s) = \frac{4\pi\alpha^2}{3s} N_{C,f} \sqrt{1 - \frac{4m_f^2}{s}} \left[\left(1 + \frac{2m_f^2}{s} \right) G_1(s) - 6\frac{m_f^2}{s} G_2(s) \right]. \quad (8)$$

Here α denotes the fine structure constant, with $\alpha = 4\pi a$, $N_{C,f}$ is the number of colors of the final state fermion, with $N_{C,f} = 1$ for colorless fermions, s is the cms energy, Ω is the spherical angle, θ the cms scattering angle, and the effective couplings $G_i(s)|_{i=1\dots 3}$ read

$$G_1(s) = Q_e^2 Q_f^2 + 2Q_e Q_f v_e v_f \text{Re}[\chi_Z(s)] + (v_e^2 + a_e^2)(v_f^2 + a_f^2)|\chi_Z(s)|^2 \quad (9)$$

$$G_2(s) = (v_e^2 + a_e^2)a_f^2|\chi_Z(s)|^2 \quad (10)$$

$$G_3(s) = 2Q_e Q_f a_e a_f \text{Re}[\chi_Z(s)] + 4v_e v_f a_e a_f |\chi_Z(s)|^2. \quad (11)$$

The reduced Z -propagator is given by

$$\chi_Z(s) = \frac{s}{s - M_Z^2 + iM_Z\Gamma_Z}, \quad (12)$$

where M_Z and Γ_Z are the mass and the width of the Z -boson and m_f is the mass of the final state fermion. $Q_{e,f}$ are the electromagnetic charges of the electron ($Q_e = -1$) and the final state fermion, resp., and the electroweak couplings v_i and a_i read

$$v_e = \frac{1}{\sin \theta_w \cos \theta_w} [I_{w,e}^3 - 2Q_e \sin^2 \theta_w] \quad (13)$$

$$a_e = \frac{1}{\sin \theta_w \cos \theta_w} I_{w,e}^3 \quad (14)$$

$$v_f = \frac{1}{\sin \theta_w \cos \theta_w} [I_{w,f}^3 - 2Q_f \sin^2 \theta_w] \quad (15)$$

$$a_f = \frac{1}{\sin \theta_w \cos \theta_w} I_{w,f}^3, \quad (16)$$

where θ_w is the weak mixing angle, and $I_{w,i}^3 = \pm 1/2$ the third component of the weak isospin for up and down particles, respectively.

The factorization mass cancels in the physical cross section in each order of the coupling constant. The initial state fermion mass dependence is solely encoded in Γ_{li} . The operator matrix elements are given by

$$\Gamma_{e^+e^+}(N) = \Gamma_{e^-e^-}(N) = \langle e | O_F^{\text{NS},S} | e \rangle \quad (17)$$

$$\Gamma_{e^+\gamma}(N) = \Gamma_{e^-\gamma}(N) = \langle \gamma | O_F^S | \gamma \rangle \quad (18)$$

$$\Gamma_{\gamma e^+}(N) = \Gamma_{\gamma e^-}(N) = \langle e | O_V^S | e \rangle, \quad (19)$$

where $O_F^{\text{NS},S}$ and O_V^S are the local twist-2 fermion and photon operators,

$$O_{F;\mu_1,\dots,\mu_N}^{\text{NS},S} = i^{N-1} \mathbf{S}[\bar{\psi} \gamma_{\mu_1} D_{\mu_2} \dots D_{\mu_N} \psi] - \text{trace terms}, \quad (20)$$

$$O_{V;\mu_1,\dots,\mu_N}^S = 2i^{N-2} \mathbf{S}[F_{\mu_1\alpha} D_{\mu_2} \dots D_{\mu_{N-1}} F_{\mu_N}^\alpha] - \text{trace terms}, \quad (21)$$

for the fermionic non-singlet (NS), singlet (S), and photonic case, [30]. Here, \mathbf{S} is the symmetrization operator of the Lorentz indices μ_1, \dots, μ_N , ψ denotes the electron field, $F_{\mu\nu}$ the

photon field–strength tensor, and $D_\mu = \partial_\mu - ieA_\mu$ the covariant derivative, with $e = \sqrt{(4\pi)^2 a}$ the electric charge and A_μ the 4–potential. We consider only one fermion species. For (17) both the flavor non–singlet (NS) and pure singlet (PS) terms contribute. It turns out that also in the present case the contributing functions both for the massive OMEs and the massless Wilson coefficients [31] can be related to nested harmonic sums [32, 33] and multiple zeta values [34]. These structures simplify further applying algebraic and structural relations [35]. One may perform the calculation in Mellin space completely and represent the result in x -space analytically resp. numerically performing the inverse Mellin transformation [35, 36].

The following representations apply

$$\Gamma_{li} \left(N, \frac{\mu^2}{m_e^2} \right) = \sum_{r=0}^{\infty} a^r(\mu^2) \sum_{n=0}^r a_{nr}(N) \ln^n \left(\frac{m_e^2}{\mu^2} \right) , \quad (22)$$

$$\tilde{\sigma}_{lk} \left(N, \frac{s'}{\mu^2} \right) = \sum_{r=0}^{\infty} a^r(\mu^2) \sum_{n=0}^r b_{nr}(N) \ln^n \left(\frac{s'}{\mu^2} \right) . \quad (23)$$

Here i, j denote the external particles in the scattering process. The coefficients a_{nr} and b_{nr} of the above series adjust such, that the physical cross section is independent of μ .

Γ_{li} , $\tilde{\sigma}_{lk}$ and the scattering cross sections σ_{ij} obey the renormalization group equations [37] :

$$\left[\left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} \right) \delta_{al} + \gamma_{al}(N, g) \right] \Gamma_{li} \left(N, \frac{\mu^2}{m_e^2}, g(\mu_e^2) \right) = 0 \quad (24)$$

$$\left[\left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} \right) \delta_{la} \delta_{kb} - \gamma_{la}(N, g) \delta_{kb} - \gamma_{kb}(N, g) \delta_{la} \right] \tilde{\sigma}_{lk} \left(\frac{s'}{m_e^2}, g(\mu^2) \right) = 0 \quad (25)$$

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} \right] \sigma_{ij} \left(\frac{s'}{\mu^2}, g(\mu^2) \right) = 0 . \quad (26)$$

The β -function $\beta(g)$ is given by

$$\beta(g) = - \sum_{k=0}^{\infty} \beta_k \frac{g^{2k+3}}{(16\pi^2)^{k+1}} , \quad (27)$$

with g the electromagnetic coupling. In QED the first expansion coefficients are [38]

$$\beta_0 = -\frac{4}{3}, \quad \beta_1 = -4 , \quad (28)$$

in the case of one light fermion. The running coupling $a(\mu^2) = g^2(\mu^2)/(16\pi^2)$ in the $\overline{\text{MS}}$ scheme is obtained as the solution of

$$\frac{da(\mu^2)}{d \ln(\mu^2)} = - \sum_{k=0}^{\infty} \beta_k a^{k+2}(\mu^2) , \quad (29)$$

which yields

$$a(\mu^2) = \frac{a_0}{1 + a_0 \beta_0 \ln(\mu^2/m_e^2)} = a_0 \left[1 + \frac{4}{3} a_0 \ln \left(\frac{\mu^2}{m_e^2} \right) \right] + O(a_0^3) , \quad (30)$$

with $a_0 = a(m_e^2)$. We rewrite the renormalization group equations replacing $\mu \partial / \partial \mu$ by $2 \partial / \partial L$, resp. $-2 \partial / \partial \lambda$, with

$$L = \ln \left(\frac{\mu^2}{m_e^2} \right) , \quad \lambda = \ln \left(\frac{s'}{\mu^2} \right) , \quad (31)$$

one obtains

$$\left[\frac{\partial}{\partial L} - \beta_0 a^2 \frac{\partial}{\partial a} + \frac{1}{2} \gamma_{ee}(N, a) \right] \Gamma_{ee} \left(N, a, \frac{\mu^2}{m_e^2} \right) + \frac{1}{2} \gamma_{e\gamma}(N, a) \Gamma_{\gamma e} \left(N, a, \frac{\mu^2}{m_e^2} \right) = 0 \quad (32)$$

$$\left[\frac{\partial}{\partial \lambda} + \beta_0 a^2 \frac{\partial}{\partial a} + \gamma_{ee}(N, a) \right] \tilde{\sigma}_{ee} \left(N, a, \frac{s'}{\mu^2} \right) + \gamma_{\gamma e}(N, a) \tilde{\sigma}_{e\gamma} \left(N, a, \frac{s'}{\mu^2} \right) = 0, \quad (33)$$

where $a = a(\mu^2)$. Here the anomalous dimensions $\gamma_{ij}(N, a)$ have the series expansion

$$\gamma_{ij}(N, a) = \sum_{k=0}^{\infty} a^{k+1} \gamma_{ij}^{(k)}. \quad (34)$$

For later convenience we also introduce the splitting functions in N -space,

$$P_{ij}^{(l)}(N) = \int_0^1 dz z^{N-1} P_{ij}^{(l)}(z) = -\gamma_{ij}^{(l)}(N). \quad (35)$$

The higher expansion coefficients of the OMEs, $\Gamma_{ij}^0, \bar{\Gamma}_{ij}^0$ and Γ_{ij}^1 , (79, 142–144), are defined such that they do not flip sign under the Mellin transform.

For the process under consideration one obtains to $O(a^2)$:

$$\begin{aligned} \Gamma_{ee} \left(N, a, \frac{\mu^2}{m_e^2} \right) &= 1 + a \left[-\frac{1}{2} \gamma_{ee}^{(0)}(N) L + \Gamma_{ee}^{(0)}(N) \right] \\ &+ a^2 \left[\left\{ \frac{1}{8} \gamma_{ee}^{(0)}(N) (\gamma_{ee}^{(0)}(N) - 2\beta_0) + \frac{1}{8} \gamma_{e\gamma}^{(0)}(N) \gamma_{\gamma e}^{(0)}(N) \right\} L^2 \right. \\ &+ \frac{1}{2} \{ -\gamma_{ee}^{(1)}(N) + 2\beta_0 \Gamma_{ee}^{(0)} - \gamma_{ee}^{(0)}(N) \Gamma_{ee}^{(0)}(N) - \gamma_{e\gamma}^{(0)}(N) \Gamma_{\gamma e}^{(0)}(N) \} L \\ &\left. + \Gamma_{ee}^{(1)} \right] + O(a^3), \end{aligned} \quad (36)$$

$$\begin{aligned} \tilde{\sigma}_{ee} \left(N, a, \frac{s'}{\mu^2} \right) &= 1 + a \left[-\gamma_{ee}^{(0)}(N) \lambda + \tilde{\sigma}_{ee}^{(0)}(N) \right] \\ &+ a^2 \left[\left\{ \frac{1}{2} \gamma_{ee}^{(0)}(N) (\gamma_{ee}^{(0)}(N) + \beta_0) + \frac{1}{4} \gamma_{e\gamma}^{(0)}(N) \gamma_{\gamma e}^{(0)}(N) \right\} \lambda^2 \right. \\ &+ \{ -\gamma_{ee}^{(1)}(N) - \beta_0 \tilde{\sigma}_{ee}^{(0)} - \gamma_{ee}^{(0)}(N) \tilde{\sigma}_{ee}^{(0)}(N) - \gamma_{\gamma e}^{(0)}(N) \tilde{\sigma}_{e\gamma}^{(0)}(N) \} \lambda \\ &\left. + \tilde{\sigma}_{ee}^{(1)} \right] + O(a^3), \end{aligned} \quad (37)$$

$$\Gamma_{\gamma e} \left(N, a, \frac{\mu^2}{m_e^2} \right) = a \left[-\frac{1}{2} \gamma_{\gamma e}^{(0)}(N) L + \Gamma_{\gamma e}^{(0)} \right] + O(a^2) \quad (38)$$

$$\tilde{\sigma}_{e\gamma} \left(N, a, \frac{s'}{\mu^2} \right) = a \left[-\frac{1}{2} \gamma_{e\gamma}^{(0)}(N) \lambda + \tilde{\sigma}_{e\gamma}^{(0)} \right] + O(a^2), \quad (39)$$

with the 1- and 2-loop splitting functions given in Refs. [39, 40]. We express the coupling constant $a = a(\mu^2)$ by (30) and assemble the differential scattering cross section (1) in terms of the three contributions: the flavor non-singlet terms with a single fermion line (I), those with an

additional closed fermion line (II), and the pure-singlet terms (III). Here we corrected Eq. (4.19, 4.22, 4.26) in Ref. [1] ⁷.

$$\begin{aligned} \frac{\widehat{d\sigma_{e^+e^-}^{\text{I}}}}{ds'} &= \frac{1}{s} \widehat{\sigma^{(0)}} \left\{ 1 + a_0 [P_{ee}^{(0)} \mathbf{L} + (\tilde{\sigma}_{ee}^{(0)} + 2\Gamma_{ee}^{(0)})] \right. \\ &\quad + a_0^2 \left\{ \frac{1}{2} P_{ee}^{(0)2} \mathbf{L}^2 + \left[P_{ee}^{(1),\text{I}} + P_{ee}^{(0)} (\tilde{\sigma}_{ee}^{(0)} + 2\Gamma_{ee}^{(0)}) \right] \mathbf{L} \right. \\ &\quad \left. \left. + (2\Gamma_{ee}^{(1),\text{I}} + \tilde{\sigma}_{ee}^{(1),\text{I}}) + 2\Gamma_{ee}^{(0)} \tilde{\sigma}_{ee}^{(0)} + \Gamma_{ee}^{(0)2} \right\} \right\} \end{aligned} \quad (40)$$

$$\frac{\widehat{d\sigma_{e^+e^-}^{\text{II}}}}{ds'} = \frac{1}{s} \widehat{\sigma^{(0)}} a_0^2 \left\{ -\frac{\beta_0}{2} P_{ee}^{(0)} \mathbf{L}^2 + \left[P_{ee}^{(1),\text{II}} - \beta_0 \tilde{\sigma}_{ee}^{(0)} \right] \mathbf{L} + (2\Gamma_{ee}^{(1),\text{II}} + \tilde{\sigma}_{ee}^{(1),\text{II}}) \right\} \quad (41)$$

$$\begin{aligned} \frac{\widehat{d\sigma_{e^+e^-}^{\text{III}}}}{ds'} &= \frac{1}{s} \widehat{\sigma^{(0)}} a_0^2 \left\{ \frac{1}{4} P_{e\gamma}^{(0)} P_{\gamma e}^{(0)} \mathbf{L}^2 + \left[P_{ee}^{(1),\text{III}} + P_{\gamma e}^{(0)} \tilde{\sigma}_{e\gamma}^{(0)} + \Gamma_{\gamma e}^{(0)} P_{e\gamma}^{(0)} \right] \mathbf{L} \right. \\ &\quad \left. + (2\Gamma_{ee}^{(1),\text{III}} + \tilde{\sigma}_{ee}^{(1),\text{III}}) + 2\tilde{\sigma}_{e\gamma}^{(0)} \Gamma_{\gamma e}^{(0)} \right\} \end{aligned} \quad (42)$$

with

$$\mathbf{L} = \hat{\mathbf{L}} + \ln(z), \quad \hat{\mathbf{L}} = \ln\left(\frac{s}{m_e^2}\right). \quad (43)$$

We will later rewrite Eqs. (40–42) in terms of $\hat{\mathbf{L}}$.

3 The Renormalization of the Operator Matrix Elements

Since the massless sub-system scattering cross sections $\tilde{\sigma}_{ab}(z, s/\mu^2)$ to $O(a_0^2)$ are known, the corresponding massive operator matrix elements for $e^\pm \rightarrow e^\pm, e^\pm \rightarrow \gamma$ and $\gamma \rightarrow e^\pm$ transitions have to be calculated. The latter two processes contribute only to first order of the coupling constant. Here, the external fermion is a massive particle, contrary to the cases studied in [26–28].

The bare OMEs are given by

$$\hat{A}_{ij}\left(\frac{m_e^2}{\mu^2}, \varepsilon, N\right) = \delta_{ij} + \sum_{k=1}^{\infty} \hat{a}^k \hat{A}_{ij}^{(k)}\left(\frac{m_e^2}{\mu^2}, \varepsilon, N\right), \quad (44)$$

with \hat{a} the unrenormalized coupling constant. The computation is performed in $D = 4 + \varepsilon$ dimensions. The electron mass is renormalized on-shell

$$p^2 = m_e^2(\mu = m_e) \quad (45)$$

with p the momentum of the external fermion. Thus renormalization concerns the wave function, the charge renormalization, and the ultraviolet singularities of the local operators. For process I counter terms emerge at $O(\hat{a}^2)$. Due to the finite fermion mass, no collinear singularities emerge.

i) Wave function renormalization.

The bare wave function ψ_0 is renormalized by

$$\psi_0 = \sqrt{Z_2(\varepsilon)} \psi. \quad (46)$$

⁷The contributions to process II have been also calculated in Ref. [25].

The Z -factors are obtained from the fermion self-energy, see Figure 1,

$$\mathbb{Z}(p, m_e) = m_e \Sigma_1(p^2, m_e) + (\not{p} - m_e) \Sigma_2(p^2, m_e) . \quad (47)$$

Expanding $\mathbb{Z}(p, m_e)$ around $p^2 = m_e^2$ one obtains

$$\frac{1}{Z_2} = 1 + 2m_e^2 \frac{\partial}{\partial p^2} \Sigma_1(p^2, m_e) \Big|_{p^2=m_e^2} + \Sigma_2(p^2, m_e) \Big|_{p^2=m_e^2} . \quad (48)$$

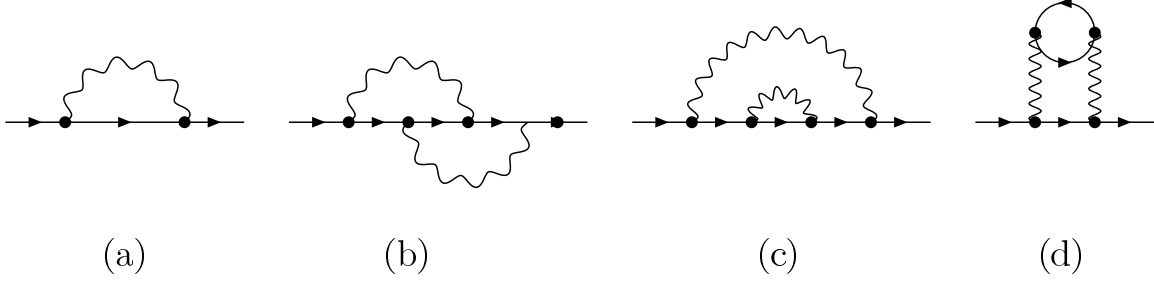


Figure 1: The self-energy diagrams.

To $O(\hat{a}^2)$ Z_2 is given by [41]

$$\begin{aligned} Z_2 &= 1 + \sum_{k=1}^{\infty} \hat{a}^k Z_2^{(k)} = 1 + \hat{a} S_{\varepsilon} \left(\frac{m_e^2}{\mu^2} \right)^{\varepsilon/2} \left[\frac{6}{\varepsilon} - 4 + \left(4 + \frac{3}{4} \zeta_2 \right) \varepsilon \right] \\ &+ \hat{a}^2 S_{\varepsilon}^2 \left(\frac{m_e^2}{\mu^2} \right)^{\varepsilon} \left\{ \left[18 \frac{1}{\varepsilon^2} - \frac{51}{2} \frac{1}{\varepsilon} + \left(\frac{433}{8} - \frac{147}{2} \zeta_2 + 96 \zeta_2 \ln(2) - 24 \zeta_3 \right) \right]_{\text{I}} \right. \\ &\left. + \left[16 \frac{1}{\varepsilon^2} - \frac{38}{3} \frac{1}{\varepsilon} + \left(\frac{1139}{18} - 28 \zeta_2 \right) \right]_{\text{II}} \right\} + O(\hat{a}^3) , \end{aligned} \quad (49)$$

with the spherical factor

$$S_{\varepsilon} = \exp \left[\frac{\varepsilon}{2} (\ln(4\pi) - \gamma_E) \right] \quad (50)$$

and ζ_k denotes Riemann's ζ -function at integer values $k \geq 2$. In the $\overline{\text{MS}}$ scheme the factors S_{ε} are set to one at the end of the calculation. In (49) we separated the terms contributing to processes I and II. At 2-loop order also counter terms (CT) contribute with the Z -factor Z_{CT} to the OME A_{ee}^{I} , which will be calculated in Section 5.

After wave function renormalization and accounting for counter terms the OME is denoted by \hat{A}_{ee} . Up to $O(\hat{a}^2)$ it is given by :

$$\begin{aligned} \hat{A}_{ee}^{\text{I}} \left(\frac{m_e^2}{\mu^2}, \varepsilon, N \right) &= 1 + \hat{a} \left[\hat{A}_{ee}^{(1),\text{I}} \left(\frac{m_e^2}{\mu^2}, \varepsilon, N \right) + Z_2^{(1)} \right] \\ &+ \hat{a}^2 \left[\hat{A}_{ee}^{(2),\text{I}} \left(\frac{m_e^2}{\mu^2}, \varepsilon, N \right) + Z_{2,\text{I}}^{(2)} + Z_{\text{CT}}^{(2)} + Z_2^{(1)} \hat{A}_{ee}^{(1),\text{I}} \left(\frac{m_e^2}{\mu^2}, \varepsilon, N \right) \right] + O(\hat{a}^3) . \\ \hat{A}_{ee}^{\text{II}} \left(\frac{m_e^2}{\mu^2}, \varepsilon, N \right) &= \hat{a}^2 \left[\hat{A}_{ee}^{(2),\text{II}} \left(\frac{m_e^2}{\mu^2}, \varepsilon, N \right) + Z_{2,\text{II}}^{(2)} \right] + O(\hat{a}^3) . \\ \hat{A}_{ee}^{\text{III}} \left(\frac{m_e^2}{\mu^2}, \varepsilon, N \right) &= \hat{a}^2 \left[\hat{A}_{ee}^{(2),\text{III}} \left(\frac{m_e^2}{\mu^2}, \varepsilon, N \right) \right] + O(\hat{a}^3) . \end{aligned} \quad (51)$$

ii) Charge renormalization.

The bare coupling \hat{a} and the renormalized coupling in the $\overline{\text{MS}}$ scheme are related by

$$\begin{aligned}\hat{a} &= Z_g^{\overline{\text{MS}^2}(\varepsilon, n_f)} a^{\overline{\text{MS}}}(\mu^2) \\ &= a^{\overline{\text{MS}}}(\mu^2) \left[1 + \delta a_1^{\overline{\text{MS}}}(\varepsilon, n_f) a^{\overline{\text{MS}}}(\mu^2) \right] + O(a^{\overline{\text{MS}^3}}),\end{aligned}\quad (52)$$

with

$$\delta a_1^{\overline{\text{MS}}}(\varepsilon, n_f) = \frac{2}{\varepsilon} \beta_0(n_f) \quad (53)$$

$$\beta_0(n_f) = -\frac{4}{3} n_f. \quad (54)$$

The above relations would apply to n_f manifestly massless fermions. Since the fermion lines are all massive the coupling constant is first being obtained in a **MOM**-scheme, which is defined by

$$\hat{\Pi}_{H,\text{BF}}(0, m_e^2) + Z_{A,H} = 0 \quad (55)$$

$$Z_g^2 = Z_{A,H}^{-1}, \quad (56)$$

cf. [27]. Here $\hat{\Pi}_{H,\text{BF}}(0, m_e^2)$ denotes the on-shell vacuum polarisation calculated using the background-field method [42],

$$\hat{\Pi}_{H,\text{BF}}^{\mu\nu}(p^2, m_e^2, \mu^2, \varepsilon, \hat{a}) = i(-g^{\mu\nu} p^2 + p^\mu p^\nu) \hat{\Pi}_{H,\text{BF}}(p^2, m_e^2, \mu^2, \varepsilon, \hat{a}) \quad (57)$$

$$\hat{\Pi}_{H,\text{BF}}(p^2, m_e^2, \mu^2, \varepsilon, \hat{a}) = \hat{a} \frac{2\beta_{0,H}}{\varepsilon} \left(\frac{m_e^2}{\mu^2} \right)^{\varepsilon/2} \exp \left(\sum_{k=2}^{\infty} \frac{\zeta_k}{k} \left(\frac{\varepsilon}{2} \right)^k \right) + O(\hat{a}^2), \quad (58)$$

with

$$\beta_{0,H} = -\frac{4}{3}. \quad (59)$$

One obtains

$$Z_g^{\text{MOM}^2} = 1 + a^{\text{MOM}}(\mu^2) \beta_{0,H} \left(\frac{m_e^2}{\mu^2} \right)^{\varepsilon/2} \exp \left(\sum_{k=2}^{\infty} \frac{\zeta_k}{k} \left(\frac{\varepsilon}{2} \right)^k \right) + O(a^{\text{MOM}}(\mu^2)^2). \quad (60)$$

Finally, we transform back to the $\overline{\text{MS}}$ scheme using

$$Z_g^{\text{MOM}^2} a^{\text{MOM}}(\mu^2) = Z_g^{\overline{\text{MS}^2}} a^{\overline{\text{MS}}}(\mu^2), \quad (61)$$

which implies

$$a^{\text{MOM}} = a^{\overline{\text{MS}}} - \beta_{0,H} \ln \left(\frac{m_e^2}{\mu^2} \right) a^{\overline{\text{MS}^2}} + O(a^{\overline{\text{MS}^3}}). \quad (62)$$

iii) Renormalization of the composite operators.

We express the inverse Z -factors in the **MOM**-scheme :

$$Z_{ij}^{-1}(a^{\text{MOM}}, n_f + 1, \mu) = \delta_{ij} - a^{\text{MOM}} \frac{\gamma_{ij}^{(0)}}{\varepsilon} + a^{\text{MOM}^2} \left[\frac{1}{\varepsilon} \left(-\frac{1}{2} \gamma_{ij}^{(1)} - \delta a_1^{\text{MOM}} \gamma_{ij}^{(0)} \right) \right]$$

$$+\frac{1}{\varepsilon^2}\left(\frac{1}{2}\gamma_{il}^{(0)}\gamma_{lj}^{(0)}+\beta_0\gamma_{ij}^{(0)}\right)\Bigg]+O(a^{\text{MOM}^3})\,, \quad (63)$$

with

$$\delta a_1^{\text{MOM}} = S_\varepsilon \frac{2\beta_{0,H}}{\varepsilon} \left(\frac{m_e^2}{\mu^2}\right)^{\varepsilon/2} \exp\left[\sum_{i=2}^{\infty} \frac{\zeta_i}{i} \left(\frac{\varepsilon}{2}\right)^i\right]. \quad (64)$$

The renormalized OMEs are given by

$$A_{ij}^{\text{MOM}} = \delta_{ij} + a^{\text{MOM}} \left[\hat{A}_{ij}^{(1)} + Z_{i,j}^{-1,(1)}\right] + a^{\text{MOM}^2} \left[\hat{A}_{ij}^{(2)} + Z_{i,j}^{-1,(2)} + Z_{i,j}^{-1,(1)} \hat{A}_{ij}^{(1)}\right] + O(a^{\text{MOM}^3}). \quad (65)$$

The transformation to the $\overline{\text{MS}}$ -scheme is obtained by (62). We split the OME into the parts I–III. The corresponding Z -factors are given by

$$\begin{aligned} [Z_{ee}^{\text{I}}(\varepsilon, N)]^{-1} &= 1 + a^{\text{MOM}} S_\varepsilon \frac{1}{\varepsilon} P_{ee}^{(0)}(N) + a^{\text{MOM}^2} S_\varepsilon^2 \left\{ \frac{1}{2\varepsilon^2} P_{ee}^{(0)2}(N) + \frac{1}{2\varepsilon} P_{ee}^{(1),\text{NS}}(N) \right\} \\ &\quad + O(a^{\text{MOM}^3}) \end{aligned} \quad (66)$$

$$\begin{aligned} [Z_{ee}^{\text{II}}(\varepsilon, N)]^{-1} &= a^{\text{MOM}^2} \left\{ -\frac{1}{2\varepsilon^2} \beta_0 P_{ee}^{(0)} + \frac{2}{\varepsilon^2} \beta_{0,H} \left(\frac{m_e^2}{\mu^2}\right)^{\varepsilon/2} \exp\left[\sum_{i=2}^{\infty} \frac{\zeta_i}{i} \left(\frac{\varepsilon}{2}\right)^i\right] + \frac{1}{2\varepsilon} P_{ee}^{(1),\text{II}} \right\} \\ &\quad + O(a^{\text{MOM}^3}) \end{aligned} \quad (67)$$

$$[Z_{ee}^{\text{III}}(\varepsilon, N)]^{-1} = a^{\text{MOM}^2} S_\varepsilon^2 \left\{ \frac{1}{2\varepsilon^2} P_{e\gamma}^{(0)}(N) P_{\gamma e}^{(0)}(N) + \frac{1}{2\varepsilon} [P_{ee}^{(1),\text{III}}(N)] \right\} + O(a^{\text{MOM}^3}) \quad (68)$$

$$[Z_{e\gamma}(\varepsilon, N)]^{-1} = a^{\text{MOM}} S_\varepsilon \frac{1}{\varepsilon} P_{e\gamma}^{(0)}(N) + O(a^{\text{MOM}^2}) \quad (69)$$

$$[Z_{\gamma e}^{\text{NS}}(\varepsilon, N)]^{-1} = a^{\text{MOM}} S_\varepsilon \frac{1}{\varepsilon} P_{\gamma e}^{(0)}(N) + O(a^{\text{MOM}^2}). \quad (70)$$

The OMEs to two-loop order after wave function and charge renormalization are given by

$$\begin{aligned} \hat{A}_{ee}^{\text{I}} &= a^{\text{MOM}} S_\varepsilon \left(\frac{m_e^2}{\mu^2}\right)^{\varepsilon/2} \left[-\frac{1}{\varepsilon} P_{ee}^{(0)} + \Gamma_{ee}^{(0)} + \varepsilon \bar{\Gamma}_{ee}^{(0)} \right] \\ &\quad + a^{\text{MOM}^2} S_\varepsilon^2 \left(\frac{m_e^2}{\mu^2}\right)^{\varepsilon} \left\{ \frac{1}{2\varepsilon^2} P_{ee}^{(0)} \otimes P_{ee}^{(0)} - \frac{1}{2\varepsilon} [P_{ee}^{(1),\text{I}} + 2\Gamma_{ee}^{(0)} \otimes P_{ee}^{(0)}] + \hat{\Gamma}_{ee}^{(1),\text{I}} \right\} \end{aligned} \quad (71)$$

$$\hat{A}_{ee}^{\text{II}} = a^{\text{MOM}^2} S_\varepsilon^2 \left(\frac{m_e^2}{\mu^2}\right)^{\varepsilon} \left\{ \frac{1}{2\varepsilon^2} 2\beta_0 P_{ee}^{(0)} - \frac{1}{2\varepsilon} [P_{ee}^{(1),\text{II}} + 4\beta_0 \Gamma_{ee}^{(0)}] + \hat{\Gamma}_{ee}^{(1),\text{II}} \right\} \quad (72)$$

$$\hat{A}_{ee}^{\text{III}} = a^{\text{MOM}^2} S_\varepsilon^2 \left(\frac{m_e^2}{\mu^2}\right)^{\varepsilon} \left\{ \frac{1}{2\varepsilon^2} P_{e\gamma}^{(0)} \otimes P_{\gamma e}^{(0)} - \frac{1}{2\varepsilon} [P_{ee}^{(1),\text{III}} + 2\Gamma_{\gamma e}^{(0)} \otimes P_{e\gamma}^{(0)}] + \hat{\Gamma}_{ee}^{(1),\text{III}} \right\}. \quad (73)$$

The renormalized OMEs in the $\overline{\text{MS}}$ -scheme are finally given by

$$\begin{aligned} A_{ee}^{\overline{\text{MS}},\text{I}}(N) &= a^{\overline{\text{MS}}} \left[-\frac{1}{2} P_{ee}^{(0)}(N) \ln \left(\frac{m_e^2}{\mu^2} \right) + \Gamma_{ee}^{(0)}(N) \right] \\ &\quad + a^{\overline{\text{MS}}} \left[\frac{1}{8} P_{ee}^{(0)2}(N) \ln^2 \left(\frac{m_e^2}{\mu^2} \right) - \frac{1}{2} [P_{ee}^{(1),\text{I}}(N) + P_{ee}^{(0)}(N) \Gamma_{ee}^{(0)}(N)] \ln \left(\frac{m_e^2}{\mu^2} \right) \right] \end{aligned}$$

$$+\hat{\Gamma}_{ee}^{(1),\text{I}}(N) + P_{ee}^{(0)}(N)\bar{\Gamma}_{ee}^{(0)}(N)\Big] + O(a^{\overline{\text{MS}}^3}) \quad (74)$$

$$\begin{aligned} A_{ee}^{\overline{\text{MS}},\text{II}}(N) &= a^{\overline{\text{MS}}^2} \left[\frac{\beta_{0,H}}{4} P_{ee}^{(0)}(N) \ln^2 \left(\frac{m_e^2}{\mu^2} \right) - \left[\frac{1}{2} P_{ee}^{(1),\text{II}}(N) + \beta_{0,H} \Gamma_{ee}^0 \right] \ln \left(\frac{m_e^2}{\mu^2} \right) \right. \\ &\quad \left. + \hat{\Gamma}_{ee}^{(1),\text{II}}(N) + 2\beta_{0,H} \bar{\Gamma}_{ee}^{(0)} \right] + O(a^{\overline{\text{MS}}^3}) \end{aligned} \quad (75)$$

$$\begin{aligned} A_{ee}^{\overline{\text{MS}},\text{III}}(N) &= a^{\overline{\text{MS}}^2} \left[\frac{1}{8} P_{e\gamma}^{(0)}(N) P_{\gamma e}^{(0)}(N) \ln^2 \left(\frac{m_e^2}{\mu^2} \right) - \frac{1}{2} \left[P_{ee}^{(1),\text{III}}(N) \right. \right. \\ &\quad \left. \left. + P_{e\gamma}^{(0)}(N) \Gamma_{\gamma e}^{(0)}(N) \right] \ln \left(\frac{m_e^2}{\mu^2} \right) + \hat{\Gamma}_{ee}^{(1),\text{III}}(N) + P_{e\gamma}^{(0)}(N) \bar{\Gamma}_{\gamma e}^{(0)}(N) \right] + O(a^{\overline{\text{MS}}^3}) . \end{aligned} \quad (76)$$

Here $\Gamma_{ij}^{(0)}$ and $\bar{\Gamma}_{ij}^{(0)}$ (90–93) denote the constant and linear term in ε of the unrenormalized one-loop massive OMEs, and $\hat{\Gamma}_{ij}^{(1)}$ (142–144) the corresponding constant part of the two-loop OMEs.

Also the OMEs $A_{e\gamma}(N)$ and $A_{\gamma e}(N)$ have to be calculated to $O(a)$. Here only the operators have to be renormalized.

$$A_{e\gamma}^{\overline{\text{MS}}}(N) = a^{\overline{\text{MS}}} \left[\hat{A}_{e\gamma}(N, \varepsilon) + Z_{e\gamma}^{-1} \right] = a^{\overline{\text{MS}}} \left[-\frac{1}{2} P_{e\gamma}^{(0)}(N) \ln \left(\frac{m_e^2}{\mu^2} \right) + \Gamma_{e\gamma}^{(0)}(N) \right] \quad (77)$$

$$A_{\gamma e}^{\overline{\text{MS}}}(N) = a^{\overline{\text{MS}}} \left[\hat{A}_{\gamma e}(N, \varepsilon) + Z_{\gamma e}^{-1} \right] = a^{\overline{\text{MS}}} \left[-\frac{1}{2} P_{\gamma e}^{(0)}(N) \ln \left(\frac{m_e^2}{\mu^2} \right) + \Gamma_{\gamma e}^{(0)}(N) \right] . \quad (78)$$

4 The $O(a)$ Operator Matrix Elements

The massive operator matrix element for process I, $A_{ee}^{(1)}$, emerges at $O(a)$ and is given by (74). In the calculation of process III to $O(a^2)$ also the OMEs $A_{e\gamma}^{(1)}$ and $A_{\gamma e}^{(1)}$, (75, 76) contribute. The Feynman rules for the operator insertions are given in Figure 2, cf. [27]. The external lines are taken on-shell, i.e. $p^2 = m_e^2$ for the fermion and $p^2 = 0$ for the photon lines, and the vector Δ is light-like with $\Delta \cdot \Delta = 0$.

In the following we present the OMEs in x -space. The unrenormalized OMEs are given by

$$\hat{A}_{ij}^{(1)}(x, \varepsilon) = \hat{a} \left(\frac{m_e^2}{\mu^2} \right)^{\varepsilon/2} S_\varepsilon \left[-\frac{1}{\varepsilon} P_{ij}^{(0)}(x) + \Gamma_{ij}^{(0)}(x) + \varepsilon \bar{\Gamma}_{ij}^{(0)}(x) + O(\varepsilon^2) \right] , \quad (79)$$

where for $\hat{A}_{ee}^{(1)\text{I}}(x, \varepsilon)$ the contributions up to $O(\varepsilon)$ are needed. In Figures 3–4 the diagrams are shown, except self-energy diagrams, which contribute to the $O(a)$ OMEs, cf. Figure 1. The expansion coefficients are the leading order splitting functions $P_{ij}^{(0)}$ [39], $\Gamma_{ij}^{(0)}$, and $\bar{\Gamma}_{ij}^{(0)}$, respectively, with

$$P_{ee}^{(0)}(x) = 8\mathcal{D}_0(x) - 4(1+x) + 6\delta(1-x) = 4 \left[\frac{1+x^2}{1-x} \right]_+ , \quad (80)$$

$$P_{e\gamma}^{(0)}(x) = 4 [x^2 + (1-x)^2] , \quad (81)$$

$$P_{\gamma e}^{(0)}(x) = 4 \left[\frac{1+(1-x)^2}{x} \right] . \quad (82)$$

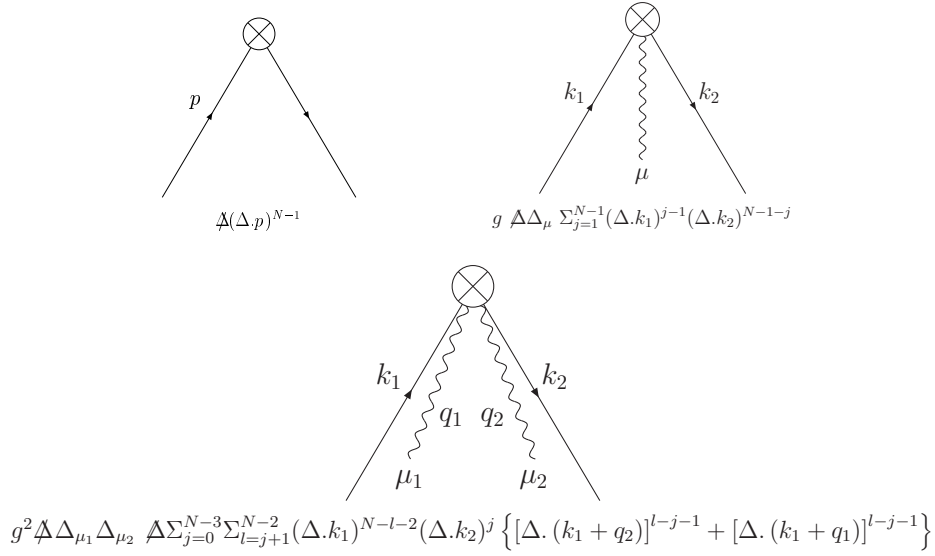


Figure 2: Feynman rules.

We define

$$\mathcal{D}_k(x) = \left(\frac{\ln^k(1-x)}{1-x} \right)_+ . \quad (83)$$

The $+$ -prescription, used to regularize some of the terms, reads

$$\int_0^1 dx [f(x)]_+ g(x) = \int_0^1 dx f(x) [g(x) - g(1)] , \quad (84)$$

and $g(x) \in \mathcal{D}[0, 1]$ denotes a test function, [43].

The splitting functions obey the well-known relations

$$P_{ee}^{(0)}(x) = P_{\gamma e}^{(0)}(1-x), \quad x < 1 \quad (85)$$

$$P_{e\gamma}^{(0)}(x) = P_{e\gamma}^{(0)}(1-x) \quad (86)$$

and

$$\int_0^1 dx P_{ee}^{(l), \text{NS}^-}(x) = 0, \quad \forall l \in \mathbb{N} \quad (87)$$

$$\int_0^1 dx x [P_{ee}^{(0)}(x) + P_{\gamma e}^{(0)}(x)] = 0 . \quad (88)$$

Eq. (87) derives from fermion number conservation with

$$P_{ee}^{(0), \text{NS}^-}(x) \equiv P_{ee}^{(0)}(x) . \quad (89)$$

Eq. (88) results from the conservation of 4-momentum.

The $O(\varepsilon^0)$ terms are

$$\begin{aligned} \Gamma_{ee}^{(0)}(x) &= -8\mathcal{D}_1(x) - 4\mathcal{D}_0(x) + 4\delta(1-x) + 2(1+x)[2\ln(1-x) + 1] \\ &= -4 \left[\frac{1+x^2}{1-x} \left\{ \ln(1-x) + \frac{1}{2} \right\} \right]_+ \end{aligned} \quad (90)$$

$$\Gamma_{e\gamma}^{(0)}(x) = 0 \quad (91)$$

$$\Gamma_{\gamma e}^{(0)}(x) = -2 \frac{1+(1-x)^2}{x} [2\ln(x) + 1] , \quad (92)$$

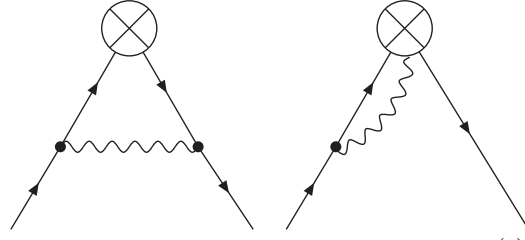


Figure 3: Diagrams contributing to $A_{ee}^{(1)}$.

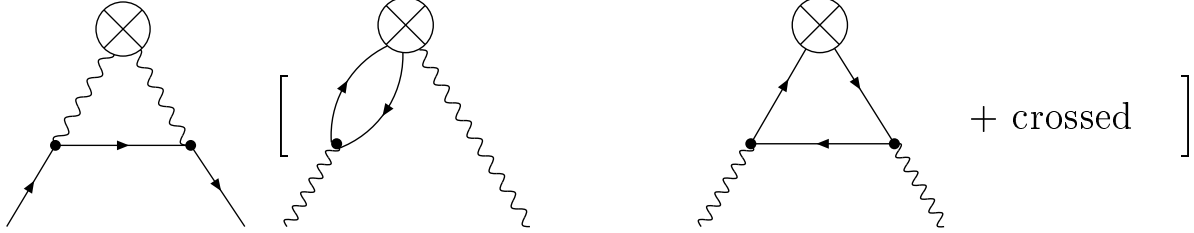


Figure 4: Diagrams contributing to $A_{\gamma e}^{(1)}$ and $A_{e\gamma}^{(1)}$.

for $\Gamma_{e\gamma}^{(0)}(x)$ cf. [18, 19]. Here we corrected Eq. (4.28c) in [1]. The linear term in ε , $\bar{\Gamma}_{ee}^{(0)}(x)$, reads

$$\begin{aligned}\bar{\Gamma}_{ee}^{(0)}(x) &= -\left\{4\mathcal{D}_2(x) + 4\mathcal{D}_1(x) + \zeta_2\mathcal{D}_0(x) + \left(4 + \frac{3}{4}\zeta_2\right)\delta(1-x)\right. \\ &\quad \left.-2(1+x)\left[\ln^2(1-x) + \ln(1-x) + \frac{1}{4}\zeta_2\right]\right\} \\ &= -2\left[\frac{1+x^2}{1-x}\left\{\ln^2(1-x) + \ln(1-x) + \frac{1}{4}\zeta_2\right\}\right]_+.\end{aligned}\quad (93)$$

In the differential cross sections (40–42) different convolutions of the expansion coefficients of the leading order OMEs and the leading order Drell-Yan scattering cross sections occur, see also Appendix A. Unlike the expansion coefficients of the massive OMEs, the coefficient functions $\sigma_{ij}^{(0)}$ are process-dependent quantities. In case of the e^+e^- annihilation process the massless Wilson coefficients can be obtained from those of the QCD Drell-Yan process given in [20, 44] to $O(a^2)$, adjusting the color factors. The $O(a)$ Wilson coefficients read ⁸:

$$\begin{aligned}\sigma_{ee}^{(0)}(x) &= 16\mathcal{D}_1(x) - 8(2 - \zeta_2)\delta(1-x) - 8\frac{\ln(x)}{1-x} - 4(1+x)[2\ln(1-x) - \ln(x)] \\ \sigma_{\gamma e}^{(0)}(x) &= \frac{1}{2}P_{e\gamma}^{(0)}(x)[2\ln(1-x) - \ln(x)] + 1 + 6x - 7x^2.\end{aligned}\quad (94)$$

The combination

$$\begin{aligned}2\Gamma_{ee}^{(0)}(x) + \sigma_{ee}^{(0)}(x) &= -8\mathcal{D}_0(x) + 8(\zeta_2 - 1)\delta(1-x) - 8\frac{\ln(x)}{1-x} + 4(1+x)(\ln(x) + 1) \\ &= -4\frac{1+x^2}{1-x}[1 + \ln(x)] - 8(1 - \zeta_2)\delta(1-x)\end{aligned}\quad (95)$$

occurs in (40–42). Here the logarithm \mathbf{L} still bears a x -dependence. Referring to $\hat{\mathbf{L}}$ instead, the $O(a_0)$ contribution to the annihilation cross section (40) is given by

$$\begin{aligned}T_{ee}^{1,\mathbf{I}} &= T_{ee}^{1,1;\mathbf{I}}\hat{\mathbf{L}} + T_{ee}^{1,0;\mathbf{I}} \\ &= \left[P_{ee}^{(0)}(x)\hat{\mathbf{L}} - P_{ee}^{(0)}(x) + 2(4\zeta_2 - 1)\delta(1-x)\right],\end{aligned}\quad (96)$$

⁸Note a typographical error in Table 6, Ref. [44].

which resembles the well-known behaviour of the splitting function $P_{ee}^{(0)}(x)$ contributing through $(\hat{\mathbf{L}} - 1)$.

5 The $O(a^2)$ operator matrix elements

In the following we discuss the $O(a_0^2)$ contributions to operator matrix elements for the processes I–III individually. Further, we investigate their contribution to the differential scattering cross sections (40–42).

5.1 The OME $A_{ee}^{(2),\text{I}}$

The Feynman diagrams contributing to $A_{ee}^{(2),\text{I}}$ are shown in Figure 5, except those contributing to the wave function renormalization, cf. Section 3. Furthermore, the counter terms shown in

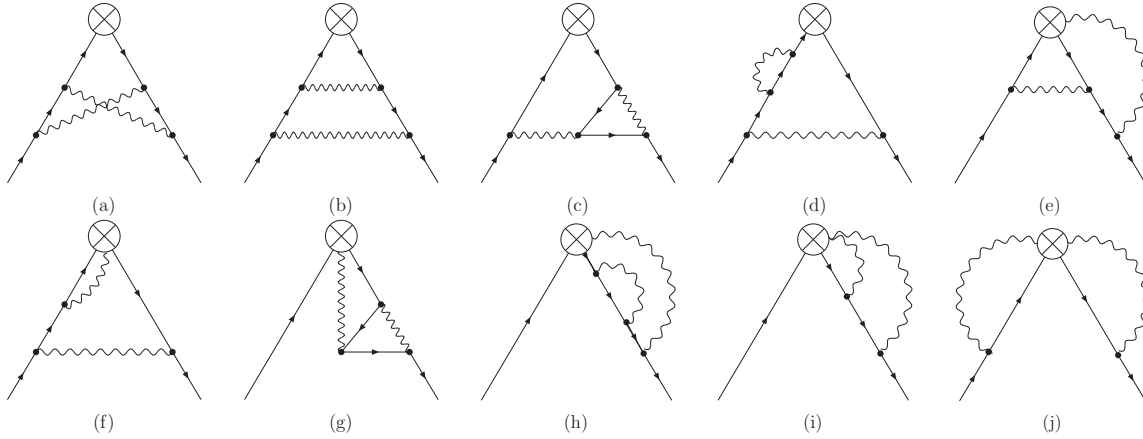


Figure 5: Feynman diagrams for the calculation of the massive two-loop operator matrix elements $A_{ee}^{(2),\text{I}}$.

Figure 6 contribute. The corresponding Z-factor is given by

$$Z_{\text{CT}}^{(2)}(x) = \left(\frac{m_e^2}{\mu^2}\right)^\varepsilon \left\{ -\frac{72}{\varepsilon^2} \delta(1-x) - \frac{24}{\varepsilon} \mathcal{D}_0(x) - 24 \mathcal{D}_1(x) + 16 \mathcal{D}_0(x) \right. \\ \left. - (64 + 18 \zeta_2) \delta(1-x) - 12 - N \left[\left(-\frac{24}{\varepsilon} - 8 \right) \delta(1-x) - 24 \mathcal{D}_0(x) \right] \right\}. \quad (97)$$

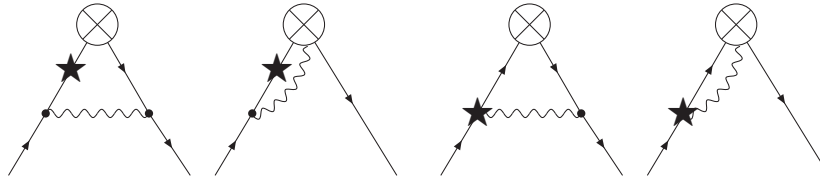


Figure 6: Counterterm diagrams. The black squares represent the counterterm vertices, cf. [45].

It contains a term $\propto N$, which cancels a corresponding contribution in $\hat{A}_{ee}^{(2),\text{I}}$. It can be rewritten, cf. [43], using

$$\int_0^1 dx N x^{N-1} \delta(1-x) = \int_0^1 dx \left(\frac{d}{dx} x^N \right) \delta(1-x) = - \int_0^1 dx x^{N-1} [x \delta'(1-x)] \quad (98)$$

and similar relations.

The two-loop diagrams shown in Figure 5 are calculated using the Feynman rules given in Figure 2. For example, the ladder diagram in Figure 5b yields :

$$\int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{\gamma_\mu(\not{k}_1 + m) \gamma_\nu(\not{k}_2 + m) \not{\Delta}(\not{k}_2 + m) \gamma^\nu(\not{k}_1 + m) \gamma^\mu}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5}} (\Delta \cdot k_2)^n, \quad (99)$$

where $n = N - 1$. We introduce the following short-hand notation for the denominators :

$$\begin{aligned} D_1 &= k_1^2 - m^2, & D_2 &= k_2^2 - m^2, & D_3 &= (k_1 - p)^2, \\ D_4 &= (k_1 - k_2)^2, & D_5 &= (k_2 - k_1 + p)^2 - m^2. \end{aligned} \quad (100)$$

The same propagator structure can be obtained for all diagrams from 5a to 5h by choosing the flow of momenta appropriately, while conveniently keeping all of the dot products coming from the Feynman rules in Figure 2 as simple as possible, i.e., as powers of $\Delta \cdot k_1$, $\Delta \cdot k_2$ and $\Delta \cdot p$. The remaining diagrams are calculated directly. The Dirac-structure of the numerators is projected multiplying by

$$\frac{1}{4} (\not{p} + m) \quad (101)$$

and taking the trace. We applied FORM [46] for this calculation. This produces a linear combination of products of all possible dot products of Δ , k_1 , k_2 and p . After canceling as much as possible these dot products against the propagators, and choosing $k_2 \cdot p$ as the only remaining irreducible numerator not involving Δ , diagrams 5a to 5d can be expressed as linear combinations of the following type of integrals

$$A_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}^{a,b} = \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{(\Delta \cdot k_1)^a (\Delta \cdot k_2)^b}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5}}, \quad (102)$$

$$B_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}^{a,b} = \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{k_2 \cdot p (\Delta \cdot k_1)^a (\Delta \cdot k_2)^b}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5}}. \quad (103)$$

Diagrams 5e and 5f can be written as linear combinations of integrals with the structure

$$E_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}^{a,b} = \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{(\Delta \cdot k_1)^a (\Delta \cdot k_2)^b}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5}} \sum_{j=0}^{n-1} (\Delta \cdot k_1)^j (\Delta \cdot k_2)^{n-1-j}, \quad (104)$$

and diagrams 5g and 5h are given by linear combinations of integrals of the form

$$F_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}^{a,b} = \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{(\Delta \cdot k_1)^a (\Delta \cdot k_2)^b}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5}} \sum_{j=0}^{n-1} (\Delta \cdot p)^j (\Delta \cdot k_1)^{n-1-j}. \quad (105)$$

We list now the results for diagrams 5a to 5h in terms of the integrals above :

Diagram 5a:

$$\begin{aligned} & \frac{1}{2} (D-2)(D-4) [A_{01111}^{1,n} + A_{11101}^{1,n} - A_{11110}^{1,n} - A_{11011}^{1,n} + 2A_{02110}^{0,1+n} - 2A_{12001}^{0,1+n} \\ & - (\Delta \cdot p) A_{01111}^{0,n}] - 2(D-4)m_e^2 [(\Delta \cdot p) A_{11111}^{0,n} - A_{11111}^{0,1+n}] + 8m_e^2 [A_{12011}^{0,1+n} + A_{12101}^{0,1+n}] \\ & + \frac{1}{2} (D-2)(D-8) [A_{11011}^{0,1+n} - A_{01111}^{0,1+n} + (\Delta \cdot p) (A_{10111}^{0,n} - A_{11101}^{0,n})] \\ & - 2(D-2) [A_{11011}^{0,1+n} - (\Delta \cdot p) A_{11011}^{0,n} - A_{11110}^{0,1+n} + 2B_{12101}^{0,1+n} + 2B_{12011}^{0,1+n}] \\ & - 4m_e^2 [A_{02111}^{0,1+n} + A_{12110}^{0,1+n}] - 16m_e^4 A_{12111}^{0,1+n} \end{aligned} \quad (106)$$

Diagram 5b:

$$(D-2)^2 [A_{11110}^{1,n} - A_{11110}^{0,1+n} - (\Delta \cdot p) A_{11110}^{0,n} - A_{02110}^{0,1+n} + 2B_{12110}^{0,1+n}] + 4(D-2)m_e^2 [A_{21110}^{0,1+n} - A_{21110}^{1,n}] - 2(D^2 - 4D + 8)m_e^2 A_{12110}^{0,1+n} - 16m_e^4 A_{22110}^{0,1+n} \quad (107)$$

Diagram 5c:

$$(D-2)(D-8) [(\Delta \cdot p) A_{01111}^{n,0} - (\Delta \cdot p) A_{11101}^{n,0} + A_{11101}^{1+n,0} + A_{11011}^{1+n,0} - A_{01111}^{1+n,0} - A_{11011}^{n,1} + A_{01111}^{n,1}] + D(D-2) A_{11110}^{1+n,0} + 8m_e^2 [2A_{21011}^{1+n,0} + 2A_{21101}^{1+n,0} - A_{21110}^{1+n,0} - A_{20111}^{1+n,0}] - (D-2)(D-4) [2A_{21001}^{1+n,0} + (\Delta \cdot p) A_{10111}^{n,0}] - 32m_e^4 A_{21111}^{1+n,0} + 4(D-2) [A_{10111}^{1+n,0} + A_{11101}^{n,1} + (\Delta \cdot p) (A_{11011}^{n,0} - A_{11110}^{n,0}) - A_{20101}^{1+n,0} - A_{11110}^{n,1} - 2B_{21101}^{1+n,0} - 2B_{21011}^{1+n,0}] + 4(D-4)m_e^2 [2A_{11111}^{1+n,0} - (\Delta \cdot p) A_{11111}^{n,0} - A_{11111}^{n,1}] \quad (108)$$

Diagram 5d:

$$(D-2)^2 [2A_{21100}^{1+n,0} - 2A_{11110}^{1+n,0} + A_{11110}^{n,1} + (\Delta \cdot p) (A_{11110}^{n,0} - A_{21100}^{n,0}) + 2B_{21110}^{1+n,0}] - 2(D-4)^2 m_e^2 A_{21110}^{1+n,0} - 4(D-2)m_e^2 [(\Delta \cdot p) A_{21110}^{n,0} + 2A_{31100}^{1+n,0} + A_{21110}^{n,1}] - 32m_e^4 A_{31110}^{1+n,0} \quad (109)$$

Diagram 5e:

$$2(D-2)(\Delta \cdot p)^2 [E_{10111}^{0,0} - E_{11101}^{0,0} + E_{01111}^{0,0}] - 2D(\Delta \cdot p) E_{10111}^{1,0} + 4E_{11011}^{0,2} + 2(D-2)(\Delta \cdot p) [E_{11011}^{1,0} + E_{11011}^{0,1} - E_{11110}^{1,0}] + 2(D-6) [E_{11011}^{1,1} - E_{01111}^{1,1}] + 2(D-4) [E_{01111}^{2,0} - E_{11011}^{2,0} + E_{11110}^{2,0} - E_{11101}^{2,0} + (\Delta \cdot p) E_{01111}^{0,1}] - 4E_{01111}^{0,2} + 4(D-3)(\Delta \cdot p) [E_{11101}^{1,0} - E_{01111}^{1,0}] + 4(\Delta \cdot p) [E_{10111}^{0,1} - E_{11101}^{0,1}] + 4E_{11110}^{1,1} - 4E_{11101}^{1,1} + 8m_e^2 [(\Delta \cdot p) (E_{11111}^{0,1} + E_{11111}^{1,0}) + E_{11111}^{1,1} - E_{11111}^{2,0}] \quad (110)$$

Diagram 5f:

$$4(D-2) [(\Delta \cdot p) E_{11110}^{0,1} - E_{11110}^{1,1}] + 16m_e^2 E_{21110}^{1,1} \quad (111)$$

Diagram 5g:

$$2(D-4) [(\Delta \cdot p) F_{01111}^{0,1} - F_{11011}^{2,0} - F_{11101}^{2,0} + F_{11110}^{2,0} + F_{01111}^{2,0}] + 2D(\Delta \cdot p) F_{10111}^{1,0} + 2(D-2) [(\Delta \cdot p) (F_{11101}^{0,1} - F_{10111}^{0,1}) + F_{11101}^{1,1} - F_{11110}^{1,1} + F_{01111}^{0,2} - F_{11011}^{0,2}] + 4(D-3) [F_{11011}^{1,1} - F_{01111}^{1,1}] + 2(D-6)(\Delta \cdot p) [F_{11101}^{1,0} - F_{01111}^{1,0}] + 4(\Delta \cdot p)^2 [F_{11101}^{0,0} - F_{10111}^{0,0} - F_{01111}^{0,0}] + 4(\Delta \cdot p) [F_{11110}^{1,0} - F_{11011}^{1,0} - F_{11011}^{0,1}] + 8(\Delta \cdot p) m_e^2 [F_{11111}^{1,0} + F_{11111}^{0,1}] - 8m_e^2 F_{11111}^{2,0} + 8m_e^2 F_{11111}^{1,1} \quad (112)$$

Diagram 5h:

$$4(D-2)(\Delta \cdot p) [F_{11110}^{0,1} - F_{11110}^{1,0} + F_{21100}^{1,0}] + 16(\Delta \cdot p) m_e^2 F_{21110}^{1,0} \quad (113)$$

Various of the integrals appearing in these expressions have only three or four propagators. The 4-propagator integrals can be represented in terms of up to three Feynman parameter integrals over the unit cube. In some cases, a direct calculation will give integrals with the following structure

$$I(\varepsilon, n) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^n f(x, y, z; \varepsilon), \quad (114)$$

while in other cases they will be of the form

$$I(\varepsilon, n) = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^n y^n f(x, y, z; \varepsilon) . \quad (115)$$

In the first case, the integrals represent a Mellin transform, and only the y and z integrals have to be performed. A mapping of Feynman parameters providing this case can be applied for (115), cf. [47], by the following transformation of the unit square into itself

$$x' = xy, \quad y' = \frac{x(1-y)}{1-xy}. \quad (116)$$

For example, the integral $A_{\nu_1 \nu_2 \nu_3 \nu_4 0}^{a,b}$ can be calculated combining first the propagators D_2 and D_4 by introducing a Feynman parameter, then performing the integral in k_2 . After this the result is combined with the remaining two propagators and the k_1 integral is carried out. One obtains

$$\begin{aligned} A_{\nu_1 \nu_2 \nu_3 \nu_4 0}^{a,b} &= C \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{b+\nu_{134}-3-\frac{\varepsilon}{2}} (1-x)^{-\nu_4+1+\frac{\varepsilon}{2}} y^{a+b+\nu_3-1} \\ &\quad \times (1-y)^{-\nu_3+1+\frac{\varepsilon}{2}} z^{\nu_{24}-3-\frac{\varepsilon}{2}} (1-z)^{\nu_1-1} \\ &\quad \times [z(1-x) + x(1-y)]^{4-\nu_{1234}+\varepsilon} . \end{aligned} \quad (117)$$

Here we use the notation

$$\nu_{i,j\dots k} = \sum_{l=\{i,j\dots k\}} \nu_l \quad (118)$$

and $C = (-1)^{\nu_{1234}} \Gamma(\nu_{1234} - 4 - \varepsilon) (m_e^2)^{4-\nu_{1234}+\varepsilon} (\Delta \cdot p)^{a+b}$. In the case $a = n, b = 0$, after interchanging x and y , the integral (117) is of the form given in (114). On the other hand, if $a = 0, b = n$, we obtain an expression of the form (115). The change of variables according to (116) yields

$$\begin{aligned} A_{\nu_1 \nu_2 \nu_3 \nu_4 0}^{0,n} &= C \int_0^1 dx' \int_0^1 dy' \int_0^1 dz x'^n (x' + y' - x'y')^{\nu_{134}-4-\varepsilon} x'^{\nu_3-1} y'^{-\nu_3+1+\frac{\varepsilon}{2}} \\ &\quad \times (1-x')^{-\nu_{1234}+7+2\varepsilon} (1-y')^{-\nu_4+1+\frac{\varepsilon}{2}} z^{\nu_{24}-3-\frac{\varepsilon}{2}} \\ &\quad \times (1-z)^{\nu_1-1} [y' + z - y'z]^{4-\nu_{1234}+\varepsilon} , \end{aligned} \quad (119)$$

which is of the form (114) as desired.

The easiest way to calculate the integrals with five propagators is to write them in terms of 4-propagator integrals, using integration by parts (IBP) identities [48]. For the 5-propagator A -type integrals which appear in expressions (106) to (109), one obtains

$$A_{11111}^{n,0} = \frac{1}{\varepsilon} (A_{12101}^{n,0} - A_{02111}^{n,0} + A_{11102}^{n,0}) \quad (120)$$

$$A_{11111}^{0,n} = \frac{1}{\varepsilon} (A_{21011}^{0,n} + A_{11120}^{0,n} + A_{01121}^{0,n} - A_{10121}^{0,n} + A_{11012}^{0,n} - A_{10112}^{0,n}) \quad (121)$$

$$A_{21111}^{n,0} = -\frac{1}{\varepsilon} \left[-A_{22101}^{n,0} - A_{21102}^{n,0} + \frac{1}{1-\varepsilon} (-A_{12102}^{n,0} + 2A_{03111}^{n,0} - 2A_{13101}^{n,0}) \right] \quad (122)$$

$$A_{12111}^{0,n} = -\frac{1}{\varepsilon} \left[-A_{12210}^{0,n} - A_{02211}^{0,n} - A_{12102}^{0,n} - A_{22101}^{0,n} + \frac{1}{1-\varepsilon} (-2A_{31101}^{0,n} + 2A_{30111}^{0,n} - A_{21210}^{0,n} + 2A_{20211}^{0,n} - A_{21102}^{0,n} - A_{21201}^{0,n} - 2A_{11310}^{0,n} + 2A_{10311}^{0,n} - 2A_{01311}^{0,n} - A_{11202}^{0,n}) \right] . \quad (123)$$

For the E -type integrals one finds

$$E_{11111}^{a,b} = \frac{1}{1-\varepsilon} \left(-bA_{11111}^{n+a,b-1} + (n+b)A_{11111}^{a,n+b-1} - E_{12101}^{a,b} + E_{02111}^{a,b} - E_{11102}^{a,b} \right) . \quad (124)$$

In this way, the 5-propagator E -type integrals can be obtained from 4-propagator integrals of the same type, together with the previously calculated $A_{11111}^{a,b}$. In (124) the factor of n multiplying the integral $A_{11111}^{a,n+b-1}$ can be absorbed in the integrand using integration by parts

$$n \int_0^1 dx x^{n-1} f(x) = \int_0^1 dx x^n [f(1)\delta(1-x) - f'(x)] , \quad (125)$$

where we have assumed that $x^n f(x)|_{x=0} = 0$ and $f(x)$ is regular at $x = 1$, which is the case for the diagrams being considered. Finally, we have

$$F_{11111}^{a,b} = \frac{1}{b+\varepsilon} \left(bF_{11111}^{a+1,b-1} + F_{12101}^{a,b} - F_{02111}^{a,b} + F_{11102}^{a,b} \right) . \quad (126)$$

This relation can be used recursively, that is, once $F_{11111}^{a,0}$ is obtained from only 4-propagator integrals, we can use it to obtain $F_{11111}^{a-1,1}$ and then $F_{11111}^{0,2}$. More details on the way to obtain these equations can be found in Appendix B.1.

The results for all of the required 4- and 5-propagator integrals, appearing in expressions (106–113) can be found in Appendix B.5. All of these integrals were checked numerically for the first few moments using **Tarcer** [49]. It turns out that all of the integrals can be expressed in terms of Nielsen integrals [50],

$$S_{n,p}(x) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 \frac{dz}{z} \ln^{n-1}(z) \ln^p(1-xz) \quad (127)$$

$$\text{Li}_n(x) = S_{n-1,1}(x) , \quad (128)$$

partly weighted by denominators $1/x$, $1/(1-x)^k$, $1/(1+x)^l$, $k, l \leq 3$, as well as the distributions $\delta(1-x)$ and $\mathcal{D}_k(x)$.

The diagrams with two photons being attached to the operator vertex are more simple, since in this case only four propagators contribute. Furthermore, three factors of Δ occur and only

one term survives in the numerator after taking the trace. For diagram 5i we obtain

$$\begin{aligned}
I_{5,i} = & S_\varepsilon^2 \int_0^1 dx x^n \left\{ \frac{32}{\varepsilon^2} \left[-\ln(1-x) - \frac{x}{1-x} \ln(x) - 1 + (1-\zeta_2)\delta(1-x) + \mathcal{D}_0(x) \right. \right. \\
& + \mathcal{D}_1(x) \left. \right] - \frac{8}{\varepsilon} \left[(4\mathcal{D}_0(x) - 4)\zeta_2 - 4 - 6\mathcal{D}_2(x) - 4\mathcal{D}_1(x) + 4\mathcal{D}_0(x) \right. \\
& + \frac{x}{1-x} (3\ln^2(x) + 2\ln(x)\ln(1-x) - 4\ln(x) - 2\text{Li}_2(1-x)) \\
& + 2\ln(1-x) + 6\ln^2(1-x) + (10 - 4\zeta_2 - 6\zeta_3)\delta(1-x) \left. \right] - 32 \\
& + 24\zeta_2 - 64\zeta_3 + 56\zeta_2\ln(1-x) + 48\ln(1-x) - 12\ln^2(1-x) \\
& - \frac{112}{3}\ln^3(1-x) + \frac{8x}{1-x} \left[\zeta_2\ln(x) - 8\text{S}_{1,2}(1-x) + 5\text{Li}_3(1-x) \right. \\
& - 3\ln(1-x)\text{Li}_2(1-x) - 2\ln^2(x)\ln(1-x) - 2\text{Li}_2(1-x) \\
& - \frac{1}{2}\ln^3(x) + 2\ln(x)\ln(1-x) + 2\ln^2(x) - 2\ln(x)\text{Li}_2(1-x) \\
& - \frac{5}{2}\ln(x)\ln^2(1-x) \left. \right] - 32\text{Li}_2(1-x) + (32 + 8\zeta_2 + 64\zeta_3)\mathcal{D}_0(x) \\
& - (56\zeta_2 + 32)\mathcal{D}_1(x) + 16\mathcal{D}_2(x) + \frac{112}{3}\mathcal{D}_3(x) \\
& \left. + (152 - 120\zeta_4 - 24\zeta_2 - 32\zeta_3)\delta(1-x) \right\} m_e^{2\varepsilon} (\Delta \cdot p)^n. \tag{129}
\end{aligned}$$

Diagram 5j yields

$$\begin{aligned}
I_{5,j} = & S_\varepsilon^2 \int_0^1 dx x^n \left\{ \frac{16}{\varepsilon^2} \left[-1 - 2\ln(1-x) - x + 2\mathcal{D}_0(x) + 2\mathcal{D}_1(x) + (1-\zeta_2)\delta(1-x) \right] \right. \\
& - \frac{16}{\varepsilon} \left[2x\ln(1-x) + 3\ln^2(1-x) - 2x - 2\zeta_2 + (2+2\zeta_2)\mathcal{D}_0(x) \right. \\
& - 2\mathcal{D}_1(x) - 3\mathcal{D}_2(x) + (2-2\zeta_3)\delta(1-x) \left. \right] + 32 - 64x - 64\zeta_3 \\
& - (20 - 12x)\zeta_2 + 16(1-2x) [\ln^2(1-x) - 2\ln(1-x)] + 16\mathcal{D}_2(x) \\
& - \frac{112}{3}\ln^3(1-x) + 56\zeta_2\ln(1-x) + (32 + 8\zeta_2 + 64\zeta_3)\mathcal{D}_0(x) \\
& - (32 + 56\zeta_2)\mathcal{D}_1(x) + \left(48 + 4\zeta_2 - \frac{23}{45}\pi^4 \right) \delta(1-x) + \frac{112}{3}\mathcal{D}_3(x) \\
& + (-1)^n \left[\frac{16}{\varepsilon^2} \left(\frac{2x}{1+x} \ln(x) + 1 - x \right) - \frac{16}{\varepsilon} \left(2x\ln(x) + 2 - 2x \right. \right. \\
& \left. \left. - \frac{x}{1+x} \left(2\zeta_2 + \ln^2(x) + 4\ln(x)\ln(1+x) + 4\text{Li}_2(-x) \right) \right) \right]
\end{aligned}$$

$$\begin{aligned}
& + 64 - 64x - 32(1+x) \ln(x) \ln(1+x) + 64x \ln(x) \\
& + \frac{8x}{1+x} \left(\zeta_2 \ln(x) + 8 \ln^2(1+x) \ln(x) + 8 \ln(x) \text{Li}_2(-x) \right. \\
& + 4 \ln^2(x) \ln(1+x) + 8 \zeta_2 \ln(1+x) + 16 \ln(1+x) \text{Li}_2(-x) \\
& + \frac{2}{3} \ln^3(x) + 16 \text{S}_{1,2}(-x) - 8 \text{Li}_3(-x) - 8 \zeta_3 \left. \right) - 16x \ln^2(x) \\
& - (12 + 20x) \zeta_2 - 32(1+x) \text{Li}_2(-x) \Big] \Big\} m_e^{2\varepsilon} (\Delta \cdot p)^n . \quad (130)
\end{aligned}$$

The result for the unrenormalized matrix element $\hat{A}_{ee}^{(2),\text{I}}$ is

$$\begin{aligned}
\hat{A}_{ee}^{(2),\text{I}} = & S_\varepsilon^2 \int_0^1 dx \, x^n \left\{ \frac{8}{\varepsilon^2} \left[-\frac{1+3x^2}{1-x} \ln(x) - 4(1+x) \ln(1-x) - 8 - 4x + 12 \mathcal{D}_0(x) \right. \right. \\
& + 8 \mathcal{D}_1(x) + (18 - 4\zeta_2) \delta(1-x) \Big] - \frac{4}{\varepsilon} \left[\frac{4x^2 + 2x - 1}{1-x} \ln(x) - 7 + 3x \right. \\
& + (1+x) \left(12 \ln^2(1-x) - \frac{1}{2} \ln^2(x) - 4 \text{Li}_2(1-x) - 8 \zeta_2 \right) - 24 \mathcal{D}_2(x) \\
& + (28 + 12x) \ln(1-x) + 2 \frac{5x^2 + 1}{1-x} \ln(x) \ln(1-x) + (16\zeta_2 - 2) \mathcal{D}_0(x) \\
& - 40 \mathcal{D}_1(x) - (10\zeta_3 - 5\zeta_2 - 24) \delta(1-x) \Big] + \left(\frac{104}{3} x + 16 + \frac{32}{1-x} \right) \zeta_2 \\
& + \frac{1+3x^2}{1-x} [6\zeta_2 \ln(x) - 8 \ln(x) \text{Li}_2(1-x) - 4 \ln^2(x) \ln(1-x)] \\
& - (24 - 18x) \ln(1-x) + 16 \frac{1+x^2}{1-x} [2 \text{Li}_3(-x) - \ln(x) \text{Li}_2(-x)] \\
& + \left(\frac{22}{3} x + 32 + \frac{64}{3(1-x)^2} - \frac{51}{1-x} - \frac{16}{3(1-x)^3} \right) \ln^2(x) + \frac{80}{3(1-x)} \\
& + 56(1+x) \zeta_2 \ln(1-x) - (104 + 32x) \ln^2(1-x) - \frac{1}{3} (1+x) \ln^3(x) \\
& + \left(\frac{178}{3} - 36x + \frac{64}{3(1-x)^2} - \frac{140}{3(1-x)} - \frac{48}{1+x} \right) \ln(x) \\
& + 4 \frac{x^2 - 8x - 6}{1-x} \ln(x) \ln(1-x) - 2 \frac{1+17x^2}{1-x} \ln(x) \ln^2(1-x) \\
& + 32 \frac{1+x}{1-x} [\ln(x) \ln(1+x) + \text{Li}_2(-x)] - \frac{112}{3} (1+x) \ln^3(1-x) \\
& + 4 \frac{5-11x^2}{1-x} [\ln(1-x) \text{Li}_2(1-x) - \text{Li}_3(1-x) - 2\zeta_3] - 30x - \frac{50}{3} \\
& - 4 \frac{13x^2 + 9}{1-x} \text{S}_{1,2}(1-x) + \frac{4(16x^2 - 10x - 27)}{3(1-x)} \text{Li}_2(1-x) \\
& + (16 - 40\zeta_2 + 128\zeta_3) \mathcal{D}_0(x) + (24 - 112\zeta_2) \mathcal{D}_1(x) + 144 \mathcal{D}_2(x)
\end{aligned}$$

$$\begin{aligned}
& + \frac{224}{3} \mathcal{D}_3(x) + \left[192 - \frac{67}{45} \pi^4 + (128 - 144 \ln(2)) \zeta_2 + 82 \zeta_3 \right] \delta(1-x) \\
& + N \left[\left(-\frac{24}{\varepsilon} - 8 \right) \delta(1-x) - 24 \mathcal{D}_0(x) \right] \\
& - (-1)^N \left[-\frac{4}{\varepsilon} \left(\frac{1+x^2}{1+x} (4 \ln(x) \ln(1+x) + 2 \zeta_2 + 4 \text{Li}_2(-x) - \ln^2(x)) \right. \right. \\
& \quad \left. \left. + 4x - 4 - 2(1+x) \ln(x) \right) + \frac{2(1-x)(45x^2 + 74x + 45)}{3(1+x)^2} \right. \\
& \quad + \frac{4(x^2 + 10x - 3)}{3(1+x)} (\zeta_2 + 2 \text{Li}_2(-x) + 2 \ln(x) \ln(1+x)) \\
& \quad + \frac{1+x^2}{1+x} \left(8 \zeta_2 \ln(x) - 24 \zeta_2 \ln(1+x) + 36 \zeta_3 - \frac{2}{3} \ln^3(x) \right. \\
& \quad - 4 \ln^2(x) \ln(1+x) - 24 \ln(x) \ln^2(1+x) - 16 \text{S}_{1,2}(1-x) \\
& \quad - 8 \ln(x) \text{Li}_2(1-x) - 48 \ln(1+x) \text{Li}_2(-x) + 40 \text{Li}_3(-x) \\
& \quad \left. \left. - 24 \ln(x) \text{Li}_2(-x) - 48 \text{S}_{1,2}(-x) \right) + 4x \frac{1-x}{(1+x)^3} \ln^2(x) \right. \\
& \quad + \frac{2(9 + 12x + 30x^2 - 20x^3 - 15x^4)}{3(1+x)^3} \ln(x) \\
& \quad - \frac{16(x^4 + 12x^3 + 12x^2 + 8x + 3)}{3(1+x)^3} \text{Li}_2(1-x) \\
& \quad \left. \left. - 4x^3 \frac{5-x}{(1+x)^3} \ln^2(x) \right] \right\} m_e^{2\varepsilon} (\Delta \cdot p)^n \tag{131}
\end{aligned}$$

Notice the presence of the term proportional to N , which is canceled by the counterterms. This term stems from diagrams with one-loop insertions, such as diagrams 5d and 5h.

5.2 The OME $A_{ee}^{(2),\text{II}}$

The matrix elements $\hat{A}_{ee}^{(2),\text{II}}$ are obtained from the fermionic one-loop insertions shown in Figure 7 supplemented by the corresponding self-energy diagrams.

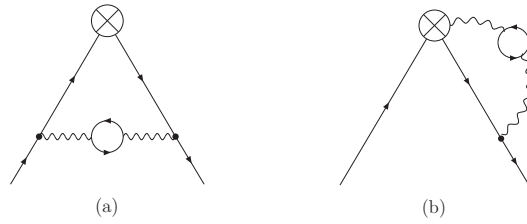


Figure 7: Feynman diagrams for the calculation of the massive two-loop operator matrix elements $A_{ee}^{(2),\text{II}}$.

One may use the fermionic one-loop vacuum polarization

$$\Pi_2^{\mu\nu}(q) = -\frac{8e^2}{(4\pi)^{D/2}} (q^2 g^{\mu\nu} - q^\mu q^\nu) \int_0^1 dx \frac{\Gamma(2-D/2) x(1-x)}{(m_e^2 - x(1-x)q^2)^{2-D/2}}, \tag{132}$$

in the corresponding one-loop diagrams, cf. Section 4. The result for $\hat{A}_{ee}^{(2),\text{II}}$ is

$$\begin{aligned}
\hat{A}_{ee}^{(2),\text{II}} = & S_\varepsilon^2 \int_0^1 dx x^n \left\{ \frac{8}{\varepsilon^2} \left[\frac{2}{3}(1+x) - \frac{4}{3}\mathcal{D}_0(x) - 3\delta(1-x) \right] - \frac{8}{\varepsilon} \left[\frac{5x-7}{9} - \frac{1+x^2}{3(1-x)} \ln(x) \right. \right. \\
& - \frac{4}{3}(1+x) \ln(1-x) + \frac{8}{3}\mathcal{D}_1(x) + \frac{2}{9}\mathcal{D}_0(x) - \left. \left(\frac{2}{3}\zeta_2 + 3 \right) \delta(1-x) \right] \\
& + \frac{76}{27}x - \frac{572}{27} - \left(12x + \frac{4}{3} + \frac{8}{1-x} \right) \ln(x) + \frac{128}{9(1-x)^2} + \frac{80}{27(1-x)} \\
& - \frac{64}{9(1-x)^3} - \frac{32}{9} \left(\frac{1}{(1-x)^2} - \frac{5}{(1-x)^3} + \frac{2}{(1-x)^4} \right) \ln(x) \\
& + \frac{16}{3}(1+x) (\ln(1-x) + \ln^2(1-x)) - \frac{2(1+x^2)}{3(1-x)} \ln^2(x) \\
& + \left(\frac{8}{3}\zeta_3 + 18\zeta_2 - \frac{4420}{81} \right) \delta(1-x) + \left(\frac{224}{27} - \frac{8}{3}\zeta_2 \right) \mathcal{D}_0(x) \\
& \left. + \frac{4}{3}(1+x)\zeta_2 - \frac{32}{3} (\mathcal{D}_1(x) + \mathcal{D}_2(x)) \right\} m_e^{2\varepsilon} (\Delta \cdot p)^n \Bigg\} . \tag{133}
\end{aligned}$$

5.3 The OME $A_{ee}^{(2),\text{III}}$

The flavor pure-singlet diagrams of Figure 8 yield $\hat{A}_{ee}^{(2),\text{III}}$. They can be calculated using the corresponding one-loop off-shell insertion of the diagrams in Figure 4, see [27].

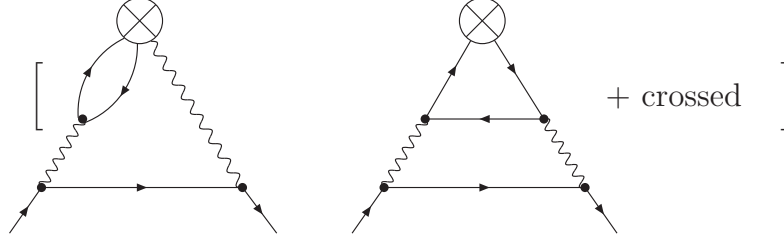


Figure 8: Feynman diagrams for the calculation of the massive two-loop operator matrix elements $A_{ee}^{(2),\text{III}}$.

For the insertion in diagrams 8a,b one obtains

$$\begin{aligned}
I_{\mu\nu}^{(8,a)}(k) = & 4 \frac{(\Delta \cdot k)^{n-1}}{(4\pi)^{D/2}} \Gamma(3-D/2) \int_0^1 dx x^{n+D/2-2} (1-x)^{-2+D/2} \left[\right. \\
& -2x(1-x) \frac{g_{\mu\nu}k^2 - 2k_\mu k_\nu}{(-k^2 + M^2)^{3-D/2}} (\Delta \cdot k)^2 - \frac{2m_e^2 g_{\mu\nu}}{(-k^2 + M^2)^{3-D/2}} (\Delta \cdot k)^2 \\
& + (1-x) \frac{2(n+1)x - n}{2-D/2} \frac{k_\mu \Delta_\nu + k_\nu \Delta_\mu}{(-k^2 + M^2)^{2-D/2}} (\Delta \cdot k) \\
& \left. + (1-x) \frac{n(1-2x) - Dx}{2-D/2} \frac{g_{\mu\nu}}{(-k^2 + M^2)^{2-D/2}} (\Delta \cdot k)^2 \right]
\end{aligned}$$

$$-(1-x)n \frac{(n+1)(1-x)-1}{(1-D/2)(2-D/2)} \frac{\Delta_\mu \Delta_\nu}{(-k^2 + M^2)^{1-D/2}} \Big] , \quad (134)$$

$$I_{\mu\nu}^{(8,b)}(k) = 8 \frac{(\Delta \cdot k)^{n-1}}{(4\pi)^{D/2}} \Gamma(2-D/2) \int_0^1 dx (x-x^2)^{D/2-1} (x^n + (1-x)^n) \\ \times \frac{(\Delta \cdot k) \Delta_\mu k_\nu - k^2 \Delta_\mu \Delta_\nu}{(-k^2 + M^2)^{2-D/2}} , \quad (135)$$

with $M^2 = m_e^2/[x(1-x)]$ and k is the momentum of the photon line. Using these results $\hat{A}_{ee}^{(2),\text{III}}$ is given by

$$\hat{A}_{ee}^{(2),\text{III}} = S_\varepsilon^2 \int_0^1 dx x^n \frac{1+(-1)^N}{2} \left\{ \frac{8}{\varepsilon^2} \left[\frac{1-x}{3x} (4x^2 + 7x + 4) + 2(1+x) \ln(x) \right] \right. \\ + \frac{4}{\varepsilon} \left[5(1+x) \ln^2(x) - \frac{1+x}{3x} (8x^2 - 17x - 16) \ln(x) + \frac{4(1-x)}{9x} (5x^2 + 23x + 14) \right] \\ + \frac{2}{x} (1-x) (4x^2 + 13x + 4) \zeta_2 + \frac{1}{3x} (8x^3 + 135x^2 + 75x + 32) \ln^2(x) \\ + \left[\frac{304}{9x} - \frac{80}{9} x^2 - \frac{32}{3} x + 108 - \frac{32}{1+x} - \frac{64(1+2x)}{3(1+x)^3} \right] \ln(x) \\ - \frac{224}{27} x^2 + 16 \frac{1-x}{3x} (x^2 + 4x + 1) [2 \ln(x) \ln(1+x) - \text{Li}_2(1-x) \\ + 2 \text{Li}_2(-x)] + (1+x) \left[4 \zeta_2 \ln(x) + \frac{14}{3} \ln^3(x) - 32 \ln(x) \text{Li}_2(-x) \right. \\ \left. - 16 \ln(x) \text{Li}_2(x) + 64 \text{Li}_3(-x) + 32 \text{Li}_3(x) + 16 \zeta_3 \right] - \frac{182}{3} x + 50 \\ \left. - \frac{32}{1+x} + \frac{800}{27x} + \frac{64}{3(1+x)^2} \right\} m^{2\varepsilon} (\Delta \cdot p)^n . \quad (136)$$

The factor $[1+(-1)^N]/2$ occurs because of the different directions of the momentum flow in the external fermion lines in the diagrams of Figure 8. For all operator matrix elements, which are not the same in the unpolarized and polarized case factors of this type appear [51]. In the present case N is even.

5.4 $O(a^2)$ Results

From the pole-terms of the unrenormalized OMEs (131, 133, 136), resp. the renormalized OMEs (74–76), one may determine the 2-loop splitting functions $P_{ee}^{(1),\text{I}}(x)$ to $P_{ee}^{(1),\text{III}}(x)$ and the constant parts of the unrenormalized OMEs $\hat{\Gamma}_{ee}^{(1),\text{I-III}}(x)$. Here the splitting functions result from a massive calculation. The operator matrix element (131) contains two branches. The contributions to the present process result from the first branch. The corresponding contribution to the 2-loop splitting function is found to be

$$P_{ee}^{(1),F}(x) \equiv \frac{1}{2} P_{ee}^{(1),\text{I}}(x) \\ = -8 \left[\frac{1+x^2}{1-x} \left\{ \ln(x) \ln(1-x) + \frac{3}{4} \ln(x) \right\} + \frac{1}{4} (1+x) \ln^2(x) + \frac{1}{4} (3+7x) \ln(x) \right. \\ \left. + \frac{5}{2} (1-x) \right] + \frac{3}{2} (1 - 8\zeta_2 + 16\zeta_3) \delta(1-x) . \quad (137)$$

The pole part of the second branch, labeled by the factor $(-1)^N$, is given by

$$P_{ee}^{(1),A}(x) \equiv P_{ee}^{(1)}(x) \equiv \frac{1}{2}\bar{P}_{ee}^{(1),I}(x) = 8 \left[\frac{1+x^2}{1+x} S_2(x) + (1+x) \ln(x) + 2(1-x) \right], \quad (138)$$

where

$$S_2(x) = -2\text{Li}_2(-x) - 2\ln(x) \ln(1+x) + \frac{1}{2} \ln^2(x) - \zeta_2, \quad (139)$$

cf. [40].

From the OMEs II and III the splitting functions

$$\begin{aligned} P_{ee}^{(1),N_f}(x) &\equiv \frac{1}{2} P_{ee}^{(1),\Pi}(x) \\ &= -\frac{80}{9} \mathcal{D}_0(x) - \frac{2}{3} (1+8\zeta_2) \delta(1-x) - \frac{16}{3} \frac{\ln(x)}{1-x} + \frac{8}{3} (1+x) \ln(x) - \frac{8}{9} (1-11x) \\ &= -8 \left[\frac{1+x^2}{1-x} \left(\frac{1}{3} \ln x + \frac{5}{9} \right) + \frac{2}{3} (1-x) \right]_+, \end{aligned} \quad (140)$$

$$\begin{aligned} P_{ee}^{(1),PS}(x) &\equiv \frac{1}{2} P_{ee}^{(1),\text{III}}(x) \\ &= 8 \left[\left(\frac{1}{2} + \frac{5}{2}x + \frac{4}{3}x^2 \right) \ln x - \frac{1}{2} (1+x) \ln^2 x + \frac{1-x}{9x} (10+x+28x^2) \right], \end{aligned} \quad (141)$$

are obtained in accordance with known results from Quantum Chromodynamics [40] setting $C_A = 0, C_F = T_F = 1$. In the present calculation we choose $N_f = 1$.

The unrenormalized operator matrix elements at the level of $\hat{A}_{ee}^{(2),\text{I-III}}$ (71–73) obey fermion number conservation, i.e. their first moment vanishes. The constant contributions to the unrenormalized OMEs (71–73) are given by

$$\begin{aligned} \hat{\Gamma}_{ee}^{(1),\text{I}} &= \frac{1+3x^2}{1-x} [6\zeta_2 \ln(x) - 8\ln(x)\text{Li}_2(1-x) - 4\ln^2(x) \ln(1-x)] \\ &+ \left(\frac{122}{3}x + 22 + \frac{32}{1-x} \right) \zeta_2 + 16 \frac{1+x^2}{1-x} [2\text{Li}_3(-x) - \ln(x)\text{Li}_2(-x)] + \frac{80}{3(1-x)} \\ &+ 56(1+x)\zeta_2 \ln(1-x) + \left(\frac{22}{3}x + 32 + \frac{64}{3(1-x)^2} - \frac{51}{1-x} - \frac{16}{3(1-x)^3} \right) \ln^2(x) \\ &- (92+20x) \ln^2(1-x) + \left(\frac{178}{3} - 36x + \frac{64}{3(1-x)^2} - \frac{140}{3(1-x)} - \frac{48}{1+x} \right) \ln(x) \\ &- \frac{1}{3} (1+x) \ln^3(x) + 4 \frac{x^2-8x-6}{1-x} \ln(x) \ln(1-x) - 2 \frac{1+17x^2}{1-x} \ln(x) \ln^2(1-x) \\ &- \frac{112}{3} (1+x) \ln^3(1-x) + 32 \frac{1+x}{1-x} [\ln(x) \ln(1+x) + \text{Li}_2(-x)] - 22x - \frac{62}{3} \\ &- 4 \frac{13x^2+9}{1-x} S_{1,2}(1-x) + 4 \frac{5-11x^2}{1-x} [\ln(1-x)\text{Li}_2(1-x) - \text{Li}_3(1-x) - 2\zeta_3] \\ &+ \frac{4(16x^2-10x-27)}{3(1-x)} \text{Li}_2(1-x) + 14(x-2) \ln(1-x) \\ &+ (16-52\zeta_2+128\zeta_3) \mathcal{D}_0(x) + (8-112\zeta_2) \mathcal{D}_1(x) + 120\mathcal{D}_2(x) + \frac{224}{3} \mathcal{D}_3(x) \end{aligned}$$

$$\begin{aligned}
& + \left[\frac{433}{8} - \frac{67}{45}\pi^4 + \left(\frac{37}{2} - 48 \ln(2) \right) \zeta_2 + 58\zeta_3 \right] \delta(1-x) \\
& + (-1)^n \left\{ \frac{2(1-x)(45x^2 + 74x + 45)}{3(1+x)^2} + \frac{2(9 + 12x + 30x^2 - 20x^3 - 15x^4)}{3(1+x)^3} \ln(x) \right. \\
& \quad + \frac{4(x^2 + 10x - 3)}{3(1+x)} (\zeta_2 + 2\text{Li}_2(-x) + 2\ln(x)\ln(1+x)) \\
& \quad + \frac{1+x^2}{1+x} \left[8\zeta_2 \ln(x) - 24\zeta_2 \ln(1+x) + 36\zeta_3 - \frac{2}{3} \ln^3(x) + 40\text{Li}_3(-x) \right. \\
& \quad - 4\ln^2(x)\ln(1+x) - 24\ln(x)\ln^2(1+x) - 24\ln(x)\text{Li}_2(-x) \\
& \quad - 48\ln(1+x)\text{Li}_2(-x) - 8\ln(x)\text{Li}_2(1-x) - 16\text{S}_{1,2}(1-x) \\
& \quad \left. - 48\text{S}_{1,2}(-x) \right] - \frac{16(x^4 + 12x^3 + 12x^2 + 8x + 3)}{3(1+x)^3} \text{Li}_2(1-x) \\
& \quad \left. + 4x \frac{1-x-5x^2+x^3}{(1+x)^3} \ln^2(x) \right\} \tag{142}
\end{aligned}$$

$$\begin{aligned}
\hat{\Gamma}_{ee}^{(1),\text{II}} &= \frac{76}{27}x - \frac{572}{27} - \left(12x + \frac{4}{3} + \frac{8}{1-x} \right) \ln(x) + \frac{128}{9(1-x)^2} + \frac{80}{27(1-x)} - \frac{64}{9(1-x)^3} \\
& - \frac{32}{9} \left(\frac{1}{(1-x)^2} - \frac{5}{(1-x)^3} + \frac{2}{(1-x)^4} \right) \ln(x) - \frac{2(1+x^2)}{3(1-x)} \ln^2(x) + \frac{4}{3}(1+x)\zeta_2 \\
& + \frac{16}{3}(1+x) (\ln(1-x) + \ln^2(1-x)) + \left(\frac{224}{27} - \frac{8}{3}\zeta_2 \right) \mathcal{D}_0(x) \\
& - \frac{32}{3} (\mathcal{D}_1(x) + \mathcal{D}_2(x)) + \left(\frac{8}{3}\zeta_3 - 10\zeta_2 + \frac{1411}{162} \right) \delta(1-x) \tag{143}
\end{aligned}$$

$$\begin{aligned}
\hat{\Gamma}_{ee}^{(1),\text{III}} &= \frac{2}{x}(1-x)(4x^2 + 13x + 4)\zeta_2 + \frac{1}{3x}(8x^3 + 135x^2 + 75x + 32)\ln^2(x) + 50 \\
& + \left[\frac{304}{9x} - \frac{80}{9}x^2 - \frac{32}{3}x + 108 - \frac{32}{1+x} - \frac{64(1+2x)}{3(1+x)^3} \right] \ln(x) - \frac{224}{27}x^2 - \frac{182}{3}x \\
& + 16\frac{1-x}{3x}(x^2 + 4x + 1) [2\ln(x)\ln(1+x) - \text{Li}_2(1-x) + 2\text{Li}_2(-x)] + \frac{800}{27x} \\
& + (1+x) \left[4\zeta_2 \ln(x) + \frac{14}{3} \ln^3(x) - 32\ln(x)\text{Li}_2(-x) - 16\ln(x)\text{Li}_2(x) \right. \\
& \left. + 64\text{Li}_3(-x) + 32\text{Li}_3(x) + 16\zeta_3 \right] - \frac{32}{1+x} + \frac{64}{3(1+x)^2} . \tag{144}
\end{aligned}$$

The constant contributions (142–144) enter the single differential cross section linearly and all contain terms of the kind $\sim 1/(1-x)^3$ and $\ln(x)/(1-x)^4$ etc. The massless Wilson coefficients of the Drell-Yan process to 2-loops are free of such terms, cf. [20, 21], like all the other Mellin convolutions which contribute. For the processes II and III non of these terms were found in [1] resp. [25], while terms like $\ln^2(x)/(1-x)^2$ are present in the cross section for process I. We therefore conclude that the use of standard matrix elements for local operators alone, (17–21), are not sufficient to reproduce the constant terms. This aspect requires further investigation.

It occurs for massive external fermion lines in contrast to the case of *massless* external parton lines. There the constant terms are correctly reproduced in the limit $s \gg m^2$.

Finally we present the $O(a_0^2)$ contributions up to the logarithmic term $\hat{\mathbf{L}}$ in (40–42), which have been computed based on the present 2-loop calculation of the massive operator matrix elements with an external massive fermion line. They are given by

$$\begin{aligned}
T_{ee}^{2,\text{Ia}} = & 16a_0^2 \left\{ \left[4\mathcal{D}_1(x) + 3\mathcal{D}_0(x) + \left(\frac{9}{8} - 2\zeta_2 \right) \delta(1-x) - 2\frac{\ln(x)}{1-x} \right. \right. \\
& - (1+x) \left(2\ln(1-x) - \frac{3}{2}(\ln(x) - 1) \right) - 1 + x \left. \right] \hat{\mathbf{L}}^2 \\
& + \left[-8\mathcal{D}_1(x) + (-7 + 4\zeta_2) \mathcal{D}_0(x) - \frac{1}{4}(11 + 7x) \ln(x) \right. \\
& + \frac{1+x^2}{1-x} \left(-\ln^2(x) + \frac{11}{4} \ln(x) + \ln(1-x) \ln(x) \right) \\
& + (1+x) \left(-2\zeta_2 + 4\ln(1-x) + \frac{1}{4} \ln^2(x) \right) + 3 + 4x \\
& \left. \left. + \left(-\frac{45}{16} + \frac{11}{2}\zeta_2 + 3\zeta_3 \right) \delta(1-x) \right] \hat{\mathbf{L}} \right\} \quad (145)
\end{aligned}$$

$$\begin{aligned}
T_{ee}^{2,\text{IIa}} = & 16a_0^2 \left\{ \left[\frac{1}{3}\mathcal{D}_0(x) - \frac{1}{6}(1+x) + \frac{1}{4}\delta(1-x) \right] \hat{\mathbf{L}}^2 \right. \\
& \left. + \left[\frac{1+x^2}{1-x} \left[\frac{2}{3}\ln(1-x) - \frac{1}{3}\ln(x) - \frac{5}{9} \right] - \frac{2}{3}(1-x) - \frac{17}{12}\delta(1-x) \right] \hat{\mathbf{L}} \right\} \quad (146)
\end{aligned}$$

$$\begin{aligned}
T_{ee}^{2,\text{IIIa}} = & 16a_0^2 \left\{ \left[\frac{1}{2}(1+x) \ln(x) + \frac{1}{3x} + \frac{1}{4} - \frac{1}{4}x - \frac{1}{3}x^2 \right] \hat{\mathbf{L}}^2 \right. \\
& + \left[(1+x) (2\ln(x) \ln(1-x) - \ln^2(x) + \text{Li}_2(1-x)) \right. \\
& + \left(\frac{4}{3x} + 1 - x - \frac{4}{3}x^2 \right) \ln(1-x) - \left(\frac{2}{3x} + 1 - \frac{1}{2}x - \frac{4}{3}x^2 \right) \ln(x) \\
& \left. \left. - \frac{8}{9x} - \frac{8}{3} + \frac{8}{3}x + \frac{8}{9}x^2 \right] \hat{\mathbf{L}} \right\} . \quad (147)
\end{aligned}$$

These terms agree with the corresponding terms in Ref. [1], Eqs. (2.30, 2.40, 2.42, 2.43) and for (146) with Ref. [25].

6 Conclusions

We have calculated the QED matrix elements with an external massive fermion for the local operators of the light-cone expansion to $O(a^2)$ up to the constant term in the dimensional parameter ε in the $\overline{\text{MS}}$ scheme. The OMEs can be renormalized applying the technique having been developed recently in [27]. The renormalized OMEs obey fermion number conservation. Various technical details of the calculation and intermediate results are provided. We investigated

the factorization of the $O(\alpha^2)$ QED initial state corrections to e^+e^- annihilation into a virtual boson for large cms energies $s \gg m_e^2$ into massive OMEs and the massless Wilson coefficients of the Drell-Yan process adapting the color coefficients to the case of QED, as being proposed in Ref. [1]. We have shown by an explicit calculation, that the representation works at one-loop order and at two-loop order including all terms up to the linear order in $\ln(s/m_e^2)$. In the constant term in $O(\alpha^2)$ the OMEs bear a few structural terms, which have not been obtained in previous direct calculations [1, 25], despite a large part of other terms appear as being expected by the factorization. Further studies are needed to reveal the reason for this. This finding appears in contrast to the case of massless external fermion and boson lines, where the corresponding cross sections have been shown to factorize including the constant terms in [18, 19, 23, 26].

Acknowledgment.

We would like to thank S. Klein, J.H. Kühn, and R. Mertig for discussions and N. Hatcher for a C++ code for `Latex` checks. This paper has been supported in part by the Alexander-von-Humboldt Foundation, by SFB-TR/9, and the EU TMR network LHCPheNet, PITN-GA-2010-264564.

A Convolutions of one-loop quantities

In Eqs. (40–42) a series of convolutions of one-loop functions occurs. Some of them were given in the Tables in [15, 52].⁹ A few more convolutions are needed which are given by :

$$\begin{aligned}\mathcal{D}_1(x) \otimes \mathcal{D}_1(x) &= \mathcal{D}_3(x) - 2\zeta_2 \mathcal{D}_1(x) + 2\zeta_3 \mathcal{D}_0(x) - \frac{\zeta_4}{4} \delta(1-x) \\ &\quad + \frac{2 [\text{Li}_3(x) - \zeta_3] - \ln(x) \text{Li}_2(x) - \ln(x) \ln^2(1-x)}{1-x}\end{aligned}\quad (148)$$

$$\begin{aligned}\mathcal{D}_1(x) \otimes \ln(1-x) &= \text{Li}_3(1-x) + 2\text{Li}_3(x) + [\ln(1-x) - \ln(x)] \text{Li}_2(x) \\ &\quad + \frac{1}{2} \ln^3(1-x) - 2\zeta_2 \ln(1-x) - \zeta_3\end{aligned}\quad (149)$$

$$\begin{aligned}\mathcal{D}_1(x) \otimes x \ln(1-x) &= x \text{Li}_3(1-x) + 2x \text{Li}_3(x) - [1+x [\ln(x) - \ln(1-x)]] \text{Li}_2(x) \\ &\quad + \frac{x}{2} \ln^3(1-x) + (1-x) \ln^2(1-x) - (1-x) \ln(1-x) \\ &\quad - x \ln(x) - (1-x) \ln(x) \ln(1-x) - 2x \zeta_2 \ln(1-x) + x(\zeta_2 - \zeta_3)\end{aligned}\quad (150)$$

$$\frac{\ln(x)}{x} \otimes x^2 = \frac{1}{9x} [3 \ln(x) + 1 - x^3] \quad (151)$$

$$\ln(x)x \otimes x^2 = x [\ln(x) + 1 - x] \quad (152)$$

$$\ln(x) \otimes x^2 = \frac{1}{4} [2 \ln(x) + 1 - x^2] \quad (153)$$

$$\frac{\ln(x)}{x} \otimes x = \frac{1}{4x} [2 \ln(x) + 1 - x^2] \quad (154)$$

$$\frac{\ln(x)}{x} \otimes 1 = \frac{1}{x} [\ln(x) + 1 - x] \quad (155)$$

$$\ln(x) \otimes \ln(1-x) = \text{Li}_3(x) - \zeta_2 \ln(x) - \zeta_3 \quad (156)$$

$$\ln(x) \otimes x \ln(1-x) = -[\text{Li}_2(x) - \zeta_2] + (1-x) [\ln(1-x) - 2] - \ln(x) \quad (157)$$

$$x \ln(x) \otimes \ln(1-x) = x [\text{Li}_2(x) - \zeta_2] - (1-x) \ln(1-x) + x \ln(x) \left[\frac{1}{2} \ln(x) - 1 \right] \quad (158)$$

$$x \ln(x) \otimes x \ln(1-x) = x [\text{Li}_3(x) - \zeta_3 - \zeta_2 \ln(x)] \quad (159)$$

$$\ln(1-x) \otimes \ln(1-x) = -\{2S_{12}(x) + 2S_{12}(1-x) + \ln(x) [\text{Li}_2(1-x) - \zeta_2] - 2\zeta_3\} \quad (160)$$

$$\begin{aligned}\ln(1-x) \otimes x \ln(1-x) &= (1-x) [\text{Li}_2(1-x) + \ln^2(1-x) - \ln(1-x)] \\ &\quad + \zeta_2(2x-1) - x [\text{Li}_2(x) + \ln(x)]\end{aligned}\quad (161)$$

$$x \ln(1-x) \otimes x \ln(1-x) = -x \{2S_{12}(x) + 2S_{12}(1-x) + \ln(x) [\text{Li}_2(1-x) - \zeta_2] - 2\zeta_3\} \quad (162)$$

We finally list some Mellin convolutions of splitting- and related functions :

$$\begin{aligned}\frac{1}{2} (P_{ee}^{(0)} \otimes P_{ee}^{(0)}) (x) &= 64\mathcal{D}_1(x) + 48\mathcal{D}_0(x) + (18 - 32\zeta_2) \delta(1-x) \\ &\quad - 32 \frac{\ln(x)}{1-x} - 16(1+x) \left[2 \ln(1-x) - \frac{3}{2} \ln(x) + \frac{3}{2} \right] - 16(1-x) \\ &= 16 \left[\frac{1+x^2}{1-x} \left\{ 2 \ln(1-x) - \ln x + \frac{3}{2} \right\} + \frac{1}{2} (1+x) \ln x - (1-x) \right]_+\end{aligned}\quad (163)$$

⁹The distribution-valued contributions to (148) were also given in [53].

$$\frac{1}{2} (P_{e\gamma}^{(0)} \otimes P_{\gamma e}^{(0)}) (x) = 16 \left[(1+x) \ln x + \frac{1}{2}(1-x) + \frac{2}{3} \frac{1}{x} (1-x^3) \right] \quad (164)$$

$$\begin{aligned} [P_{ee}^{(0)} \otimes \Gamma_{ee}^{(0)}] (x) &= -96\mathcal{D}_2(x) - 112\mathcal{D}_1(x) + 8(1+8\zeta_2)\mathcal{D}_0(x) \\ &\quad + 8(-8\zeta_3 + 4\zeta_2 + 3)\delta(1-x) + 16(1+x)\text{Li}_2(x) \\ &\quad + 48(1+x) (\ln^2(1-x) - \zeta_2) + 8(11+3x) \ln(1-x) \\ &\quad - 4(1+x) + 32 \left\{ \frac{1+x^2}{1-x} \left[\ln(1-x) + \frac{1}{2} \right] - \frac{1}{4}(1-x) \right\} \ln(x) \end{aligned} \quad (165)$$

$$\begin{aligned} [\Gamma_{ee}^{(0)} \otimes \Gamma_{ee}^{(0)}] (x) &= 64\mathcal{D}_3(x) + 96\mathcal{D}_2(x) - 32(1+4\zeta_2)\mathcal{D}_1(x) \\ &\quad - 32(1+2\zeta_2-4\zeta_3)\mathcal{D}_0(x) \\ &\quad + 16(1-\zeta_2+4\zeta_3-\zeta_4)\delta(1-x) + 8(3+x) \\ &\quad + 64 \left[1 + \frac{\ln(x)}{1-x} + (1+x) \ln(1-x) - \frac{3}{4}(1+x) \ln(x) \right] \zeta_2 \\ &\quad - 64(1+x)\zeta_3 + 16(1+x) \ln(1-x) - 32(1+x) \ln^3(1-x) \\ &\quad - 16(5+x) \ln^2(1-x) + 48(1+x) \ln(x) \ln^2(1-x) \\ &\quad + 16(3+x) \ln(x) \ln(1-x) + 4(3+7x) \ln(x) \\ &\quad - 16 \frac{\ln(x)}{1-x} [4 \ln^2(1-x) + 4 \ln(1-x) + 1] \\ &\quad + 16 \left[2(1+x) \ln(1-x) + 3(1+x) \ln(x) - (1-x) \right. \\ &\quad \left. - 4 \frac{\ln(x)}{1-x} \right] \text{Li}_2(1-x) - 32(1+x) \text{Li}_3(1-x) \\ &\quad + 32 \left[3(1+x) - \frac{4}{1-x} \right] S_{1,2}(1-x) \end{aligned} \quad (166)$$

$$\begin{aligned} [P_{e\gamma}^{(0)} \otimes \Gamma_{\gamma e}^{(0)}] (x) &= -16(1+x) \ln^2(x) - \left(\frac{64}{3x} + 48 + 32x \right) \ln(x) - \frac{304}{9x} (1-x^3) \\ &\quad - 8(1-x) . \end{aligned} \quad (167)$$

$$\begin{aligned} [P_{ee}^{(0)} \otimes (2\Gamma_{ee}^{(0)} + \tilde{\sigma}_{ee}^{(0)})] (x) &= -128\mathcal{D}_1(x) - 16(7-4\zeta_2)\mathcal{D}_0(x) - 16(3-7\zeta_2)\delta(1-x) \\ &\quad - 8 \left(\frac{1+x^2}{1-x} \right) \ln(x) [4 \ln(1-x) - 2 \ln(x) - 1] \\ &\quad - 8(1+x) [\ln^2(x) - 8 \ln(1-x) + 4\zeta_2] \\ &\quad - 32x \ln(x) + 88 + 24x \end{aligned} \quad (168)$$

$$\begin{aligned} [\Gamma_{ee}^{(0)} \otimes \tilde{\sigma}_{ee}^{(0)}] (x) &= -128\mathcal{D}_3(x) - 96\mathcal{D}_2(x) + 192(1+\zeta_2)\mathcal{D}_1(x) \\ &\quad + 32(2+\zeta_2-8\zeta_3)\mathcal{D}_0(x) + 32(-2+\zeta_2-2\zeta_3+\zeta_4)\delta(1-x) \\ &\quad - 16(1+3x) + 8(3-x) \ln(x) - 64(2+x) \ln(1-x) \\ &\quad - 112(1+x) \ln(x) \ln^2(1-x) + 64(1+x) \ln^3(1-x) \\ &\quad + 24(1+x) \ln^2(x) \ln(1-x) + 4(3+x) \ln^2(x) \\ &\quad - 16(5-x) \ln(x) \ln(1-x) + 4(28-4x) \ln^2(1-x) \\ &\quad + 16 \frac{\ln(x)}{1-x} \left[10 \ln^2(1-x) - 2 \ln(x) \ln(1-x) \right. \end{aligned}$$

$$\begin{aligned}
& +6 \ln(1-x) - \ln(x) - 2 \Big] + 128(1+x)\zeta_3 \\
& +16 \left[2 + 4 \frac{\ln(x)}{1-x} - (1+x)(3 \ln(x) + 4 \ln(1-x)) \right] \text{Li}_2(1-x) \\
& +64(1+x) \text{Li}_3(1-x) - 16 \left[7(1+x) - \frac{8}{1-x} \right] S_{1,2}(1-x) \\
& + \left[96(1+x)(\ln(x) - \ln(1-x)) + 48x - 128 \frac{\ln(x)}{1-x} - 80 \right] \zeta_2 \quad (169)
\end{aligned}$$

$$\begin{aligned}
\left[\bar{\Gamma}_{\gamma e}^{(0)} \otimes P_{e\gamma}^{(0)} \right] (x) &= 8 \left\{ \frac{2}{3}(1+x) \ln^3(x) + (3+2x) \ln^2(x) + \left[(7x+8) + \frac{1}{2}(1+x)\zeta_2 \right] \ln(x) \right. \\
&\quad \left. - (4x^2 + 7x - 11) + \frac{3}{4}(1-x)\zeta_2 \right\} \quad (170)
\end{aligned}$$

$$\begin{aligned}
\left[P_{\gamma e}^{(0)} \otimes \tilde{\sigma}_{e\gamma}^{(0)} \right] (x) &= -\frac{40}{3}x + \frac{272}{9}x^2 + \frac{16}{9}\frac{1}{x} - \frac{56}{3} \\
&+ \left(16 + \frac{64}{3}\frac{1}{x} - \frac{64}{3}x^2 - 16x \right) \ln(1-x) \\
&+ (32 + 32x) [\text{Li}_2(1-x) + \ln(x) \ln(1-x)] \\
&+ \left(16 + \frac{32}{3}x^2 + 8x \right) \ln(x) - 8(1+x) \ln^2(x) \quad (171)
\end{aligned}$$

$$\begin{aligned}
\left[\Gamma_{\gamma e}^{(0)} \otimes \tilde{\sigma}_{e\gamma}^{(0)} \right] (x) &= \frac{8}{3}(1+x) \ln^3(x) + 8x \ln^2(x) \\
&+ \left[-\frac{16}{9}\frac{1}{x} - 20 + 28x - \frac{152}{9}x^2 - 32(1+x)\zeta_2 \right] \ln(x) \\
&+ \left[-\frac{304}{9}\frac{1}{x} - 8 + 8x \frac{304}{9}x^2 \right] \ln(1-x) \\
&+ \left(32x + 48 + \frac{64}{3}\frac{1}{x} \right) [\text{Li}_2(x) - \zeta_2] + (32 + 32x) [\text{Li}_3(x) - \zeta_3] \\
&+ \frac{28}{27}\frac{1}{x} + \frac{314}{9} + \frac{232}{9}x - \frac{1666}{27}x^2 \quad (172)
\end{aligned}$$

$$\begin{aligned}
\bar{\Gamma}_{ee}^{(0)}(x) \otimes P_{ee}^{(0)}(x) &= \frac{128}{3} \mathcal{D}_3(x) + 72 \mathcal{D}_2(x) + (24 - 48\zeta_2) \mathcal{D}_1(x) + (64\zeta_3 - 20\zeta_2 + 32) \mathcal{D}_0(x) \\
&+ 4(1+x) \left(4S_{1,2}(x) + 6\zeta_2 \ln(1-x) - 12\zeta_3 - \frac{16}{3} \ln^3(1-x) - 3 \ln(1-x) \right) \\
&- 2 \frac{1+3x^2}{1-x} \zeta_2 \ln(x) - 16 \frac{1+x^2}{1-x} \ln(x) [\ln(1-x) + \ln^2(1-x)] \\
&- 8(1-x) \text{Li}_2(x) - (52 + 20x) \ln^2(1-x) + (14 + 6x)\zeta_2 - 8x \ln(x) \\
&- 24 - 8x + \left(\frac{9}{2}\zeta_2 + 32\zeta_3 - 84\zeta_4 + 24 \right) \delta(1-x) \quad (173)
\end{aligned}$$

B Integrals

In the following we present some decompositions of five-propagator integrals in terms of four-propagator integrals using integration by parts [48]. In some cases Mellin–Barnes representations are applied for checks. In a sample calculation we illustrate main steps of the computation of the massive OMEs. Furthermore, we list a series of complicated integrals and finally summarize the structure of the individual diagrams in terms of Nielsen integrals.

B.1 Integration by parts

Eqs. (120–123) are obtained from the integration by parts relation

$$\mathcal{E}_{\nu_1\nu_2\nu_3\nu_4\nu_5}^{a,b}(q, l) = \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{\partial}{\partial q^\mu} \left(l^\mu \frac{(\Delta \cdot k_1)^a (\Delta \cdot k_2)^b}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5}} \right) = 0 , \quad (174)$$

where $q^\mu = k_1^\mu, k_2^\mu$ and $l^\mu = p^\mu, k_1^\mu, k_2^\mu$. For example, (120) results subtracting $\mathcal{E}_{11111}^{n,0}(k_2, k_1)$ from $\mathcal{E}_{11111}^{n,0}(k_2, k_2)$. On the other hand, the most complicated equation of this family, namely, (123) can be obtained from

$$\begin{aligned} & \mathcal{E}_{21111}^{0,n}(k_1, k_2) - \mathcal{E}_{21111}^{0,n}(k_1, k_1) + \mathcal{E}_{11211}^{0,n}(k_1, k_2) - \mathcal{E}_{11211}^{0,n}(k_1, k_1) \\ & - (D - 5) [\mathcal{E}_{12111}^{0,n}(k_1, k_2) - \mathcal{E}_{12111}^{0,n}(k_1, k_1)] = 0 . \end{aligned} \quad (175)$$

In the same way, it is possible to obtain relations for $A_{11111}^{n,1}$ and $A_{11111}^{1,n}$, which are needed to obtain the E -type integrals from (124).

Sometimes it is even possible to use (174) to reduce 4-propagator A -type integrals to a linear combination of integrals with just three propagators. For example, it can be shown that

$$A_{31110}^{n,0} = \frac{1}{1 + \varepsilon} \left\{ A_{32100}^{n,0} - \frac{2}{\varepsilon} \left[A_{23100}^{n,0} + \frac{3}{1 - \varepsilon} (A_{14100}^{n,0} - A_{04110}^{n,0}) \right] \right\} . \quad (176)$$

It turns out to be much easier to calculate $A_{31110}^{n,0}$ from this equation than by introducing Feynman parameters directly.

Equations for the B -type integrals can also be obtained by setting one of the powers of the propagators in (174) to zero. For example, from

$$\mathcal{E}_{11110}^{0,n}(k_1, p) + \mathcal{E}_{11110}^{0,n}(k_2, p) = 0 , \quad (177)$$

one may obtain

$$B_{12110}^{0,n} = \frac{n}{2} (\Delta \cdot p) A_{11110}^{0,n-1} - m^2 A_{21110}^{0,n} - \frac{1}{2} A_{01210}^{0,n} + \frac{1}{2} A_{21010}^{0,n} , \quad (178)$$

and from $\mathcal{E}_{11101}^{n,0}(k_1, k_2) = 0$, one may get

$$\begin{aligned} B_{21101}^{n,0} &= \frac{1}{2} \left[n A_{11101}^{n-1,1} - A_{21001}^{n,0} + A_{21100}^{n,0} - A_{20101}^{n,0} + A_{11200}^{n,0} - A_{10201}^{n,0} \right. \\ & \quad \left. - A_{11002}^{n,0} + A_{10102}^{n,0} + 2m^2 A_{11102}^{n,0} \right] . \end{aligned} \quad (179)$$

These kind of equations can be used to check the results obtained directly using Feynman parameters for this type of integrals.

Let us consider now the E -type integrals. Using

$$\sum_{j=0}^{n-1} (\Delta \cdot k_2)^j (\Delta \cdot k_1)^{n-1-j} = \frac{(\Delta \cdot k_2)^n - (\Delta \cdot k_1)^n}{\Delta \cdot k_2 - \Delta \cdot k_1} , \quad (180)$$

we can rewrite them as

$$E_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}^{a,b} = J_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}^{a,n+b} - J_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}^{n+a,b} , \quad (181)$$

where

$$J_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}^{a,b} = \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{(\Delta \cdot k_1)^a (\Delta \cdot k_2)^b}{D_\Delta D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5}} , \quad (182)$$

with $D_\Delta = \Delta \cdot k_2 - \Delta \cdot k_1$. We define also

$$K_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}^{a,b} = \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{(\Delta \cdot k_1)^a (\Delta \cdot k_2)^b}{D_\Delta^2 D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5}} . \quad (183)$$

Then, from

$$\int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{\partial}{\partial k_2^\mu} \left((k_2 - k_1)^\mu \frac{(\Delta \cdot k_1)^a (\Delta \cdot k_2)^b}{D_\Delta D_1 D_2 D_3 D_4 D_5} \right) = 0 , \quad (184)$$

and using

$$K_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}^{a+1,b} - K_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}^{a,b+1} = J_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}^{a,b} , \quad (185)$$

and

$$J_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}^{a+1,b} - J_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}^{a,b+1} = A_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}^{a,b} , \quad (186)$$

Eq. (124) is obtained. Similar manipulations lead to (126).

B.2 Mellin-Barnes representation

The 5-propagator integrals can be also calculated and/or checked using a Mellin-Barnes representation [54]. To do this, we follow the ideas developed in [55], where a Mellin-Barnes representation was used to study the massless two-loop two-point function with arbitrary powers of the propagators. In fact, these ideas have been already applied to the calculation of massive operator matrix elements with external gluon lines [56].

The presence of the mass and the Mellin variable n make our integrals more complicated compared with the massless two-point function, so in order to obtain the Mellin-Barnes representation of lowest dimensionality, it is necessary to choose the right momenta flow in the diagrams. Let us consider the integrals $A_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5}^{a,b}$. It turns out that changing variables

$$k_1 \rightarrow k_2 + p , \quad k_2 \rightarrow k_1 , \quad (187)$$

so that now the propagators in (100) become

$$\begin{aligned} D_1 &= (k_2 + p)^2 - m^2 , & D_2 &= k_1^2 - m^2 , & D_3 &= k_2^2 , \\ D_4 &= (k_2 - k_1 + p)^2 , & D_5 &= (k_1 - k_2)^2 - m^2 , \end{aligned}$$

one can obtain a two-dimensional Mellin-Barnes representation. This is achieved combining the propagators D_2 and D_5 with a Feynman parameter, and then combining the result with D_4 introducing another Feynman parameter. After completing squares and performing the k_1 integral, one obtains

$$A_{\nu_1, \nu_2, \nu_3, \nu_4, \nu_5}^{a,b} = \int_0^1 dx \int_0^1 dy \int \frac{d^D k_2}{(2\pi)^D} i \frac{\Gamma(\nu_{245} - 2 - \varepsilon/2)}{\Gamma(\nu_2) \Gamma(\nu_4) \Gamma(\nu_5)} \frac{(-1)^{\nu_{245}}}{(4\pi)^{D/2}}$$

$$\begin{aligned}
& \times x^{\nu_2-1} (1-x)^{\nu_5-1} y^{-\nu_4+1-\frac{\varepsilon}{2}} (1-y)^{\nu_4-1} \\
& \times \frac{(\Delta \cdot k_2 + \Delta \cdot p)^a [(1-xy)\Delta \cdot k_2 + (1-y)\Delta \cdot p]^b}{D_1^{\nu_1} D_3^{\nu_3} [ym_e^2 - x(1-y)D_1 - xy(1-x)D_3]^{\nu_{245}-D/2}}.
\end{aligned} \tag{188}$$

Now, the denominator can be split twice using

$$\frac{1}{(X+Y)^\nu} = \frac{Y^{-\nu}}{2\pi i \Gamma(\nu)} \int_{-i\infty}^{+i\infty} d\sigma X^\sigma Y^{-\sigma} \Gamma(-\sigma) \Gamma(\sigma + \nu), \tag{189}$$

and we are left with only two propagators, which can be calculated directly. One obtains

$$\begin{aligned}
A_{\nu_1, \nu_2, \nu_3, \nu_4, \nu_5}^{a,b} & \propto \frac{(-1)^{\nu_{1\dots 5}+1}}{\Gamma(\nu_2)\Gamma(\nu_4)\Gamma(\nu_5)} \sum_{k=0}^b (-1)^k \frac{b!}{(b-k)!k!} \frac{1}{(2\pi i)^2} \\
& \times \int_{-i\infty}^{+i\infty} d\sigma \int_{-i\infty}^{+i\infty} d\tau \Gamma(-\sigma) \Gamma(-\tau) \frac{\Gamma(\sigma+k+\nu_5) \Gamma(\sigma+\tau+\nu_2)}{\Gamma(2\sigma+\tau+k+\nu_{25})} \\
& \times \Gamma(\sigma+\tau+\nu_{245}-2-\varepsilon/2) \frac{\Gamma(-\sigma-\tau+\nu_{13}-2-\varepsilon/2)}{\Gamma(\nu_3-\sigma) \Gamma(\nu_1-\tau)} \\
& \times \frac{\Gamma(2\sigma+\tau+k-\nu_1-2\nu_3+4+\varepsilon) \Gamma(-\sigma+a+b-k+\nu_3)}{\Gamma(\sigma+\tau+a+b-\nu_{13}+4+\varepsilon)} \\
& \times \frac{\Gamma(\tau+b-k+\nu_4) \Gamma(-\tau-\nu_{25}-2\nu_4+k+4+\varepsilon)}{\Gamma(b-\nu_{245}+4+\varepsilon)}.
\end{aligned} \tag{190}$$

For this expression, we can use the package **MB** [57] to check the integrals numerically, which can be done for up to relatively large values of n . This is the main advantage of this method over **Tarcer** [49], which allows to check only the first three or four moments¹⁰. One may try also to obtain analytic results starting from equation (190), but unfortunately, unlike the case where we have an external massless particle [56], this turns out to be rather complicated, although being possible in some cases.

This method can also be applied to check the 4-propagator integrals. For example, the denominators in (117) and (119) can be split using equation (189) just once, leading to a simple one-dimensional Mellin-Barnes representation. In spite of this simplicity, on occasions the **MB** package cannot find a proper contour. This usually happens for integrals with high powers of the propagators, like the 4-propagator integrals appearing in equations (122) and (123), which have a highly singular structure. As it is well-known, this problem can be cured introducing an additional regularization parameter.

B.3 Sample calculations

As it was mentioned before, the method of integration by parts has the advantage that it allows to write all 5-propagator integrals in terms of 4-propagator ones, which are simpler to calculate because they can be parameterized using only three Feynman parameters. In the case of Eq. (123), this simplicity is somewhat spoiled by the high powers of the propagators in the 4-propagator integrals. Let us consider one of the integrals appearing in that equation, namely,

$$A_{11310}^{0,n} \propto \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{n+2} (1-x)^{-3+2\varepsilon} (x+y-xy)^{1-\varepsilon}$$

¹⁰**Tarcer** can be modified to handle larger values of n on the expense of very long computational times.

$$\times y^{-2+\varepsilon/2}(1-y)^{\varepsilon/2}z^{-1-\varepsilon/2}(z+y-zy)^{-2+\varepsilon} . \quad (191)$$

It is not difficult to see that the singularity structure in ε is shared by the three Feynman parameters in a way that is not easy to disentangle. If one tries to expand in ε directly, the integrals will be ill-defined, and it is difficult to find suitable subtraction integrals to cure this. To solve the problem, we decomposed the integral as

$$A_{11310}^{0,n} \propto I_1 + I_2 - I_3 , \quad (192)$$

where

$$\begin{aligned} I_1 &= -\frac{2}{\varepsilon} \int_0^1 dx \int_0^1 dy x^{n+2}(1-x)^{-3+2\varepsilon}(x+y-xy)^{1-\varepsilon} \\ &\quad \times y^{-4+\varepsilon/2}(1-y)^{\varepsilon/2} , \end{aligned} \quad (193)$$

$$\begin{aligned} I_2 &= \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{n+2}(1-x)^{-3+2\varepsilon}(x+y-xy)^{1-\varepsilon} \\ &\quad \times y^{-4+\varepsilon/2}(1-y)^{1+\varepsilon/2}z^{-\varepsilon/2}(z+y-zy)^{-1+\varepsilon} , \end{aligned} \quad (194)$$

$$\begin{aligned} I_3 &= \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{n+2}(1-x)^{-3+2\varepsilon}(x+y-xy)^{1-\varepsilon} \\ &\quad \times y^{-3+\varepsilon/2}(1-y)^{1+\varepsilon/2}z^{-\varepsilon/2}(z+y-zy)^{-2+\varepsilon} . \end{aligned} \quad (195)$$

Here, integral I_1 was obtained performing integration by parts in z . It turns out that the change of variables (116), which was used before to write the integrals as Mellin transforms, can be used now to obtain expressions that can be safely expanded in ε . For example, for the integrals I_2 and I_3 , we change variables by

$$y = y'z' , \quad z = \frac{y'(1-z')}{1-z'y'} , \quad (196)$$

which leads to

$$\begin{aligned} I_2 &= \int_0^1 dx \int_0^1 dy' \int_0^1 dz' x^{n+3-\varepsilon}(1-x)^{-3+2\varepsilon} \left(1 + \frac{1-x}{x}y'z'\right)^{1-\varepsilon} \\ &\quad \times y'^{-4+\varepsilon}z'^{-4+\varepsilon/2}(1-z')^{-\varepsilon/2}(1-y'z')^\varepsilon , \end{aligned} \quad (197)$$

$$\begin{aligned} I_3 &= \int_0^1 dx \int_0^1 dy' \int_0^1 dz' x^{n+3-\varepsilon}(1-x)^{-3+2\varepsilon} \left(1 + \frac{1-x}{x}y'z'\right)^{1-\varepsilon} \\ &\quad \times y'^{-4+\varepsilon}z'^{-3+\varepsilon/2}(1-z')^{-\varepsilon/2}(1-y'z')^\varepsilon . \end{aligned} \quad (198)$$

Now, the expansion in ε (not including the term $(1-x)^{-3+2\varepsilon}$) will produce only logarithms that are regular at $x = 1$, and can be calculated using

$$\begin{aligned} \int_0^1 dy y^{-a+b\varepsilon}(1-y)^{c\varepsilon} \ln^k(1+\chi y) &= \frac{1}{1-a+b\varepsilon} \left[-k\chi \int_0^1 dy \frac{y^{-a+1+b\varepsilon}(1-y)^{c\varepsilon}}{1+\chi y} \ln^{k-1}(1+\chi y) \right. \\ &\quad \left. + c\varepsilon \int_0^1 dy y^{-a+1+b\varepsilon}(1-y)^{-1+c\varepsilon} \ln^k(1+\chi y) \right] , \end{aligned} \quad (199)$$

recursively to analytically continue the expressions by reducing the highly singular powers in the integration variables. A few other integrals which appear in (123) can be performed in the same way.

The E - and F -type integrals can be written in terms of Feynman parameters from the corresponding expressions for the A -type integrals using

$$E_{\nu_1\nu_2\nu_3\nu_4\nu_5}^{a,b} = \sum_{j=0}^{n-1} A_{\nu_1\nu_2\nu_3\nu_4\nu_5}^{a+j,b+n-1-j} , \quad (200)$$

$$F_{\nu_1\nu_2\nu_3\nu_4\nu_5}^{a,b} = \sum_{j=0}^{n-1} (\Delta \cdot p)^{n-1-j} A_{\nu_1\nu_2\nu_3\nu_4\nu_5}^{a+j,b} . \quad (201)$$

For example,

$$\begin{aligned} E_{11101}^{1,0} &\propto \int_0^1 dx \int_0^1 dy \int_0^1 dz \sum_{j=0}^{n-1} x^{n-1-j-\varepsilon/2} (1-x)^{-\varepsilon/2} y^{j+1-\varepsilon/2} (1-y)^{n-1-j} \\ &\quad \times z^{-1-\varepsilon/2} [yz + x(1-x)(1-y)^2]^\varepsilon \\ &= \int_0^1 dx \int_0^1 dy \int_0^1 dz \left[x^{n+1-\varepsilon/2} \frac{y^{-\varepsilon/2} (1-y)^{-\varepsilon/2} z^{-1-\varepsilon/2}}{(x+y-xy)[xz + y(1-y)(1-x)^2]^{-\varepsilon}} \right. \\ &\quad \left. - x^n \frac{(x+y-xy)^{-1-\varepsilon} (1-x)^2 y^{1-\varepsilon/2} (1-y)^{-\varepsilon/2} z^{-1-\varepsilon/2}}{(x^2+y-x^2y)[yz + x^2(1-y)]^{-\varepsilon}} \right] , \end{aligned} \quad (202)$$

and

$$\begin{aligned} E_{21110}^{1,1} &\propto \int_0^1 dx \int_0^1 dy \int_0^1 dz \sum_{j=0}^{n-1} x^{n+1-j-\varepsilon/2} (1-x)^{\varepsilon/2} y^{\varepsilon/2} (1-y)^{n+1} \\ &\quad \times z^{-1-\varepsilon/2} (1-z)[z(1-x) + xy]^{-1+\varepsilon} \\ &= I_a - I_b , \end{aligned} \quad (203)$$

where

$$I_a = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{n+1} \frac{y^{-1+\varepsilon/2} (1-y)^{2-\varepsilon/2} z^{-1-\varepsilon/2} (1-z)}{(1-x)^{-\varepsilon/2} [zy + (1-y)(1-x)]^{1-\varepsilon}} , \quad (204)$$

and

$$I_b = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{n+1} \frac{y^{\varepsilon/2} (1-y)^{-1+\varepsilon/2} z^{-1-\varepsilon/2} (1-z)}{(1-x)^{1-2\varepsilon} (x+y-xy)^\varepsilon (y+z-yz)^{1-\varepsilon}} . \quad (205)$$

Integral I_a is particularly difficult. One way to perform it is to use the integral representation of the hypergeometric function [58]

$${}_2F_1(\alpha, \beta; \gamma; z) = \frac{1}{B(\beta, \gamma - \beta)} \int_0^1 dt t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-tz)^{-\alpha} , \quad (206)$$

for $\text{Re } \gamma > \text{Re } \beta > 0$ and then use the following analytic continuation [59]

$${}_2F_1(\alpha, \beta; \gamma; z) = (1-z)^{-\beta} {}_2F_1\left(\beta, \gamma - \alpha; \gamma; \frac{z}{1-z}\right) , \quad (207)$$

to write

$$I_a = I_{a1} + I_{a2} - I_{a3} , \quad (208)$$

with

$$I_{a1} = K \int_0^1 dx \int_0^1 dy \int_0^x dz x^{n+1} \frac{(1-x)^{-1+\varepsilon} y^{-1+\varepsilon/2} z^{\varepsilon/2} (1-z)^{-\varepsilon}}{[1-x+(x-z)y]^{-\varepsilon/2}} , \quad (209)$$

$$I_{a2} = K \int_0^1 dx \int_0^1 dy \int_x^1 dz x^{n+1} \frac{(1-x)^{-1+\varepsilon} y^{-1+\varepsilon/2} z^{\varepsilon/2} (1-z)^{-\varepsilon}}{[1-x+(x-z)y]^{-\varepsilon/2}} , \quad (210)$$

$$I_{a3} = K \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{n+1} \frac{(1-x)^{-1+\varepsilon} y^{\varepsilon/2} z^{\varepsilon/2} (1-z)^{-\varepsilon}}{[1-x(1-y)-yz]^{-\varepsilon/2}} , \quad (211)$$

where $K = -\frac{2}{\varepsilon} + \varepsilon\zeta_2 + \frac{\varepsilon^2}{2}\zeta_3 + O(\varepsilon^3)$. The integral I_{a3} can be performed straightforwardly. For integral I_{a1} we change variables $z = xz'$, while for I_{a2} one substitutes $z = (1-x)z' + x$, which leads to

$$I_{a1} = K \int_0^1 dx \int_0^1 dy \int_0^1 dz' x^{n+2+\varepsilon/2} \frac{(1-x)^{-1+\varepsilon} y^{-1+\varepsilon/2} z'^{\varepsilon/2} (1-xz')^{-\varepsilon}}{[1-x+(1-z')xy]^{-\varepsilon/2}} , \quad (212)$$

$$I_{a2} = K \int_0^1 dx \int_0^1 dy \int_0^1 dz' x^{n+1} \frac{(1-x)^{\varepsilon/2} y^{-1+\varepsilon/2} (1-z')^{-\varepsilon}}{[(1-x)z' + x]^{-\varepsilon/2} [1-z'y]^{-\varepsilon/2}} . \quad (213)$$

Now integral I_{a2} is easy to obtain, and integral I_{a1} can be done using another analytic continuation of the hypergeometric function, namely

$$\begin{aligned} {}_2F_1(\alpha, \beta; \gamma; z) &= \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)} \left(-\frac{1}{z}\right)^\alpha {}_2F_1\left(\alpha, \alpha+1-\gamma; \alpha+1-\beta; \frac{1}{z}\right) \\ &\quad + \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)} \left(-\frac{1}{z}\right)^\beta {}_2F_1\left(\beta, \beta+1-\gamma; \beta+1-\alpha; \frac{1}{z}\right) \end{aligned} \quad (214)$$

which yields

$$\begin{aligned} I_{a1} &= K \int_0^1 dx \int_0^1 dy \int_0^1 dz \left[-x^{n+2+\varepsilon} (1-x)^{-1+\varepsilon} y^{-1-\varepsilon} z^{\varepsilon/2} (1-z)^{\varepsilon/2} \right. \\ &\quad \times (1-xz)^{-\varepsilon} \left(1 + \frac{1-x}{(1-z)x} y \right)^{\varepsilon/2} \\ &\quad + \left(\frac{1}{\varepsilon} + \frac{\varepsilon}{2}\zeta_2 + \frac{\varepsilon^2}{4}\zeta_3 \right) x^{n+2} (1-x)^{-1+2\varepsilon} \\ &\quad \left. \times z^{\varepsilon/2} (1-z)^{-\varepsilon/2} (1-xz)^{-\varepsilon} \right] . \end{aligned} \quad (215)$$

This integral can now be computed using known integrals and specific integrals given in the following Section.

B.4 Polylogarithmic integrals and analytic continuations

In the following we list a series of integrals over polylogarithms derived in the present calculation beyond those which were given in [60]. They may be of use in other higher order calculations.

$$\begin{aligned} \int_0^1 dy \frac{1-x}{1-(1-x)y} \text{Li}_2\left(-\frac{1-y}{x^2y}\right) &= 2\zeta_2 \ln(x) + \frac{7}{6} \ln^3(x) + 4 \ln(x) \text{Li}_2(-x) \\ &\quad + 3 \ln(x) \text{Li}_2(x) + 2 \text{Li}_3(x) - 2 \text{Li}_3(x^2) , \end{aligned} \quad (216)$$

$$\int_0^1 dy \frac{x}{(1 - (1 - x)y)^2} \text{Li}_2 \left(-\frac{1 - y}{x^2 y} \right) = -\text{Li}_2(1 - x) - \frac{1}{2} \ln^2(x) - \zeta_2, \quad (217)$$

$$\begin{aligned} \int_0^1 dy \frac{x}{1 - xy} \text{Li}_2 \left(-\frac{1 - x}{x^2 y(1 - y)} \right) &= 2\text{S}_{1,2}(x) + 4\text{S}_{1,2}(1 - x) - 4\zeta_3 \\ &\quad - \ln(x) \ln^2(1 - x), \end{aligned} \quad (218)$$

$$\begin{aligned} \int_0^1 dy \text{Li}_2 \left(\frac{x}{(1 - y + xy)(x + y - xy)} \right) &= \frac{1 + x}{1 - x} [\text{Li}_2(x^2) + 2 \ln(x) \ln(1 + x)] \\ &\quad - 4 \ln(1 - x) - \frac{2x}{1 - x} [\zeta_2 + 2 \ln(x)], \end{aligned} \quad (219)$$

$$\begin{aligned} \int_0^1 dy \frac{x}{1 - xy} \ln(1 - y) \ln(1 - xy) &= \text{Li}_3(1 - x) + \frac{1}{2} \ln(x) \ln^2(1 - x) \\ &\quad - \frac{1}{3} \ln^3(1 - x) - \zeta_2 \ln(1 - x) - \zeta_3, \end{aligned} \quad (220)$$

$$\int_0^1 dy \frac{x(1 - x)}{(1 - xy)^2} \ln(1 - y) \ln(1 - xy) = \text{Li}_2(x) + \ln^2(1 - x) + \ln(1 - x), \quad (221)$$

$$\int_0^1 dy \frac{x}{1 - xy} \ln(y) \ln(1 - xy) = \text{S}_{1,2}(x), \quad (222)$$

$$\int_0^1 dy \frac{x}{(1 - xy)^2} \ln(y) \ln(1 - xy) = \text{Li}_2(x) + \frac{1}{2} \ln^2(1 - x) + \ln(1 - x). \quad (223)$$

In all of the results given above, it is assumed that $0 \leq x \leq 1$.

Some double integrals that were also used are shown below, where $u = (1 - x)/x$,

$$\begin{aligned} \int_0^1 dy \int_0^1 dz \frac{z \ln(1 - zy) \ln(1 + uyz)}{1 - zy} &= 2 + \frac{1 + u}{u} \left[\frac{1}{2} \ln^2(1 + u) - \ln(1 + u) + \text{Li}_2(-u) \right] \\ &= 2 + \frac{\ln(x) - \text{Li}_2(1 - x)}{1 - x} \end{aligned} \quad (224)$$

$$\begin{aligned} \int_0^1 dy \int_0^1 dz \frac{z \ln(1 - z) \ln(1 + uyz)}{1 - zy} &= 3 + \frac{1 + u}{u} \left[\frac{1}{2} \ln^2(1 + u) - 2 \ln(1 + u) + \text{Li}_2(-u) \right] \\ &= 3 + \frac{2 \ln(x) - \text{Li}_2(1 - x)}{1 - x} \end{aligned} \quad (225)$$

$$\begin{aligned} \int_0^1 dy \int_0^1 dz \frac{yz^2 \ln^2(1 + uyz)}{1 - zy} &= \frac{1}{4} - \frac{3}{2u} + \frac{1}{2} \left(1 - \frac{1}{u^2} \right) \ln^2(1 + u) \\ &\quad - \left(\frac{1}{2} - \frac{1}{u} - \frac{3}{2u^2} \right) \ln(1 + u) \\ &= \frac{1 - 2x}{2(1 - x)^2} \ln^2(x) + \frac{1 - 4x}{2(1 - x)^2} \ln(x) - \frac{7x - 1}{4(1 - x)} \end{aligned} \quad (226)$$

$$\begin{aligned} \int_0^1 dy \int_0^1 dz \frac{yz^2 \ln(z) \ln(1 + uyz)}{1 - zy} &= -\frac{5}{4u} + \frac{5}{2} - 2\text{S}_{1,2}(-u) + \text{Li}_3(-u) \\ &\quad + \left[-\frac{1}{2u^2} + \frac{1}{u} - \ln(1 + u) \right] \text{Li}_2(-u) \\ &\quad + \left[\zeta_2 - \frac{7}{4} + \frac{3}{4u^2} - \frac{1}{u} \right] \ln(1 + u) \\ &= -\frac{5x}{4(1 - x)} + \frac{5}{2} - \zeta_3 - \ln(x) \text{Li}_2(x) \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2}\ln^2(x)\ln(1-x) + \text{Li}_3(x) - \text{Li}_3(1-x) \\
& + \left(\frac{x}{1-x} - \frac{3x^2}{4(1-x)^2} + \frac{7}{4} - \zeta_2 \right) \ln(x) \\
& + \frac{3x^2-2x}{2(1-x)^2} \text{Li}_2(1-x) + \frac{3x^2-2x}{4(1-x)^2} \ln^2(x) \quad (227)
\end{aligned}$$

$$\begin{aligned}
\int_0^1 dy \int_0^1 dz \frac{z \ln(z) \ln(1+uyz)}{1-zy} &= 3 - \ln(1+u) \text{Li}_2(-u) + \ln(1+u) \zeta_2 + \frac{1}{u} \text{Li}_2(-u) \\
& - 2 \left(1 + \frac{1}{u} \right) \ln(1+u) + \text{Li}_3(-u) - 2S_{1,2}(-u) \\
& = 3 + \left(\frac{2}{1-x} - \zeta_2 \right) \ln(x) - \frac{1}{2} \frac{x}{1-x} \ln^2(x) \\
& - \frac{x}{1-x} \text{Li}_2(1-x) - \text{Li}_3(1-x) - S_{1,2}(1-x) \quad (228)
\end{aligned}$$

$$\begin{aligned}
\int_0^1 dy \int_0^1 dz \frac{z \ln(y) \ln(1+uyz)}{1-zy} &= -1 + \left(1 + \frac{1}{u} - \zeta_2 \right) \ln(1+u) + \ln(1+u) \text{Li}_2(-u) \\
& - \text{Li}_3(-u) + 2S_{1,2}(-u) \\
& = -1 + \left(\zeta_2 - \frac{1}{1-x} \right) \ln(x) \\
& + \text{Li}_3(1-x) + S_{1,2}(1-x) \quad (229)
\end{aligned}$$

$$\begin{aligned}
\int_0^1 dy \int_0^1 dz \frac{z \ln^2(1+uyz)}{1-zy} &= 2 + \left(1 + \frac{1}{u} \right) [\ln^2(1+u) - 2\ln(1+u)] \\
& = 2 + \frac{\ln^2(x)}{1-x} + 2 \frac{\ln(x)}{1-x} \quad (230)
\end{aligned}$$

$$\begin{aligned}
\int_0^1 dy \int_0^1 dz \frac{yz^3 \ln(z) \ln(1-xy)}{1-zy} &= -\frac{1}{9} + \frac{5}{4}x - \frac{1}{2} \left(x - \frac{1}{2}x^2 \right) \zeta_2 + \frac{1}{2}x^2 \zeta_3 \\
& + \left(\frac{3}{4}x - \frac{23}{36} - \frac{1}{9x} \right) \ln(1-x) - \frac{1}{4} (1-x^2) \text{Li}_2(x) \\
& - \frac{1}{2}x^2 [\text{Li}_3(x) + S_{1,2}(x)] - \frac{1}{8} (1-x^2) \ln^2(1-x) \\
& + \frac{1}{2} (1-x^2) \ln(1-x) \zeta_2 \quad (231)
\end{aligned}$$

$$\begin{aligned}
\int_0^1 dy \int_0^1 dz \frac{z^3 \ln(1-xy) \ln(y)}{1-zy} &= -\frac{5}{24}x - \frac{4}{9}x^2 + \frac{1}{3}x^3 \zeta_3 + \left(\frac{1}{3}x^2 + \frac{1}{6}x - \frac{x^3}{9} \right) \zeta_2 \\
& + \frac{1}{3} (1-x^3) \ln(1-x) [\text{Li}_2(x) - \zeta_2] \\
& + \left(-\frac{x^2}{9} - \frac{x}{18} + \frac{1}{6} \right) \ln(1-x) + \frac{1}{3} (2-x^3) S_{1,2}(x) \\
& - \frac{1}{3} (1-2x^3) \text{Li}_3(x) + \frac{1}{18} (1-x^3) \ln^2(1-x) \\
& - \frac{1}{3} \left(x^2 + \frac{x}{2} + \frac{x^3}{3} \right) \text{Li}_2(x) \quad (232)
\end{aligned}$$

$$\begin{aligned}
\int_0^1 dy \int_0^1 dz \frac{z^3 \ln(1-y) \ln(1-xy)}{1-zy} &= \frac{x}{3} + \frac{x}{2}(1+x) \zeta_2 + \frac{2}{3}x^3 \zeta_3 - \frac{1}{6}(1-x) \ln(1-x) \\
& - \frac{1}{3} (1-2x^3) \text{Li}_3(x) + \frac{2}{3} (1-x^3) \ln(1-x) \text{Li}_2(x)
\end{aligned}$$

$$\begin{aligned}
& + \left(1 - \frac{2}{3}x^3\right) S_{1,2}(x) + \frac{x}{6}(1-x)\text{Li}_2(x) \\
& + \frac{x}{12}(1-x)\ln^2(1-x) + \frac{1}{9}(1-x^3)\ln^3(1-x) \quad (233)
\end{aligned}$$

$$\begin{aligned}
\int_0^1 dy \int_0^1 dz \frac{yz^3 \ln(1-xy) \ln(y)}{1-zy} = & -\frac{2}{3} - \frac{3}{4}x + \frac{1}{2}x^2\zeta_3 - \frac{1}{2}(1-x^2)\ln(1-x)\zeta_2 \\
& + \left(\frac{7}{12} - \frac{x}{4} - \frac{1}{3x}\right)\ln(1-x) + \left(\frac{x}{2} - \frac{x^2}{4}\right)\zeta_2 \\
& - \left(\frac{x}{2} - \frac{1}{3x} + \frac{x^2}{4}\right)\text{Li}_2(x) + \frac{1}{8}(1-x^2)\ln^2(1-x) \\
& + \left(x^2 - \frac{1}{2}\right)\text{Li}_3(x) + \left(1 - \frac{x^2}{2}\right)S_{1,2}(x) \\
& + \frac{1}{2}(1-x^2)\ln(1-x)\text{Li}_2(x) \quad (234)
\end{aligned}$$

$$\begin{aligned}
\int_0^1 dy \int_0^1 dz \frac{yz^3 \ln^2(1-xy)}{1-zy} = & -\frac{2}{3} + x^2\zeta_2 + (1-x^2)\text{Li}_2(x)\ln(1-x) \\
& + \frac{2}{3}\left(1 - \frac{1}{x}\right)\ln(1-x) + S_{1,2}(x) + x^2\text{Li}_3(x) \\
& + x(x-1)\text{Li}_2(x) + \left(\frac{1}{6} - x + \frac{1}{3x} + \frac{x^2}{2}\right)\ln^2(1-x) \\
& + \frac{1}{3}(1-x^2)\ln^3(1-x) \quad (235)
\end{aligned}$$

$$\begin{aligned}
\int_0^1 dy \int_0^1 dz \frac{z^3 \ln(z) \ln(1-xy)}{1-zy} = & \frac{5x}{12} + \frac{7x^2}{9} + \left(\frac{x^3}{9} - \frac{x^2}{3} - \frac{x}{6}\right)\zeta_2 + \frac{x^3}{3}\zeta_3 \\
& + \frac{1}{3}(1-x^3)\ln(1-x)\zeta_2 - \frac{x^3}{3}[S_{1,2}(x) + \text{Li}_3(x)] \\
& - \frac{1}{9}(1-x^3)\text{Li}_2(x) - \frac{1}{18}(1-x^3)\ln^2(1-x) \\
& + \left(-\frac{7}{12} + \frac{5}{36}x + \frac{4}{9}x^2\right)\ln(1-x) \quad (236)
\end{aligned}$$

$$\begin{aligned}
\int_0^1 dy \int_0^1 dz \frac{z^3 \ln^2(1-xy)}{1-zy} = & \frac{x^2}{3} + x^3\zeta_2 + \frac{2}{3}(1-x^3)\ln(1-x)\text{Li}_2(x) \\
& - \frac{x}{3}(1-x)\ln(1-x) + \frac{2}{9}(1-x^3)\ln^3(1-x) \\
& + \left(-\frac{x}{3} - \frac{2}{3}x^2 + x^3\right)\text{Li}_2(x) + \frac{2}{3}x^3\text{Li}_3(x) + \frac{2}{3}S_{1,2}(x) \\
& + \left(\frac{1}{2} - \frac{x}{3} - \frac{2}{3}x^2 + \frac{1}{2}x^3\right)\ln^2(1-x) \quad (237)
\end{aligned}$$

$$\begin{aligned}
\int_0^1 dy \int_0^1 dz \frac{z^3 \ln(1-zy) \ln(1-xy)}{1-zy} = & \frac{3x}{8} + \frac{11x^2}{18} + \left(-\frac{3}{4} + \frac{5x}{36} + \frac{11x^2}{18}\right)\ln(1-x) \\
& + \left(-\frac{1}{9} - \frac{x}{6} - \frac{x^2}{3} + \frac{11x^3}{18}\right)\text{Li}_2(x) + \frac{11}{18}x^3\zeta_2 \\
& - \frac{1}{3}(1-x^3)[\text{Li}_3(x) - S_{1,2}(x)] + \frac{x^3}{3}\zeta_3 \\
& + (1-x^3)\left[\frac{1}{18}\ln^3(1-x) + \frac{1}{3}\ln(1-x)\text{Li}_2(x)\right]
\end{aligned}$$

$$\int_0^1 dy \int_0^1 dz \frac{yz^3 \ln(1-zy) \ln(1-xy)}{1-zy} = -\frac{7}{9} + \frac{11x}{12} + \left(\frac{11}{12}x - \frac{17}{36} - \frac{4}{9x} \right) \ln(1-x) \quad (238)$$

$$\begin{aligned} & + \left(-\frac{1}{18} - \frac{x}{12} - \frac{x^2}{6} + \frac{11x^3}{36} \right) \ln^2(1-x) \\ & + \left(\frac{11}{24}x^2 - \frac{3}{8} - \frac{x}{4} + \frac{1}{6x} \right) \ln^2(1-x) \\ & + \frac{11}{12}x^2\zeta_2 + \frac{1}{12}(1-x^2)\ln^3(1-x) \\ & + \left(\frac{1}{3x} - \frac{3}{4} - \frac{x}{2} + \frac{11}{12}x^2 \right) \text{Li}_2(x) + \frac{1}{2}x^2\zeta_3 \\ & + \frac{1}{2}(1-x^2)[S_{1,2}(x) - \text{Li}_3(x)] \\ & + \frac{1}{2}(1-x^2)[+\ln(1-x)\text{Li}_2(x)] \end{aligned} \quad (239)$$

$$\begin{aligned} \int_0^1 dy \int_0^1 dz \frac{z^2 \ln(z) \ln(1-xy)}{1-zy} &= \frac{5}{4}x - \frac{x}{2} \left(1 - \frac{x}{2} \right) \zeta_2 + \frac{x^2}{2}\zeta_3 \\ &- \frac{3}{4}(1-x)\ln(1-x) - \frac{1}{4}(1-x^2)\text{Li}_2(x) \\ &+ \frac{1}{2}(1-x^2)\ln(1-x)\zeta_2 - \frac{1}{8}(1-x^2)\ln^2(1-x) \\ &- \frac{1}{2}x^2[\text{Li}_3(x) + S_{1,2}(x)] \end{aligned} \quad (240)$$

$$\begin{aligned} \int_0^1 dy \int_0^1 dz \frac{z^2 \ln(1-xy) \ln(y)}{1-zy} &= -\frac{3}{4}x - \frac{1}{2}(1-x^2)\ln(1-x)\zeta_2 \\ &+ \frac{1}{8}(1-x^2)\ln^2(1-x) + \frac{x}{2} \left(1 - \frac{x}{2} \right) \zeta_2 \\ &+ \frac{1}{4}(1-x)\ln(1-x) - \frac{x}{2} \left(1 + \frac{x}{2} \right) \text{Li}_2(x) \\ &+ \left(1 - \frac{x^2}{2} \right) S_{1,2}(x) + \left(x^2 - \frac{1}{2} \right) \text{Li}_3(x) + \frac{1}{2}x^2\zeta_3 \\ &+ \frac{1}{2}(1-x^2)\ln(1-x)\text{Li}_2(x) \end{aligned} \quad (241)$$

$$\begin{aligned} \int_0^1 dy \int_0^1 dz \frac{z^2 \ln(1-y) \ln(1-xy)}{1-zy} &= x\zeta_2 + x^2\zeta_3 + \left(\frac{3}{2} - x^2 \right) S_{1,2}(x) \\ &+ (1-x^2)\ln(1-x)\text{Li}_2(x) - \left(\frac{1}{2} - x^2 \right) \text{Li}_3(x) \\ &+ \frac{1}{6}(1-x^2)\ln^3(1-x) \end{aligned} \quad (242)$$

$$\begin{aligned} \int_0^1 dy \int_0^1 dz \frac{z^2 \ln^2(1-xy)}{1-zy} &= x^2\zeta_2 + (1-x^2)\ln(1-x)\text{Li}_2(x) + S_{1,2}(x) + x^2\text{Li}_3(x) \\ &- x(1-x)\text{Li}_2(x) + \left(\frac{1}{2}x^2 + \frac{1}{2} - x \right) \ln^2(1-x) \\ &+ \frac{1}{3}(1-x^2)\ln^3(1-x) \end{aligned} \quad (243)$$

$$\int_0^1 dy \int_0^1 dz \frac{z^2 \ln(1-zy) \ln(1-xy)}{1-zy} = \frac{3}{4}x + \left(\frac{3}{8}x^2 - \frac{1}{4}x - \frac{1}{8} \right) \ln^2(1-x)$$

$$\begin{aligned}
& -\frac{3}{4}(1-x)\ln(1-x) + \frac{1}{12}(1-x^2)\ln^3(1-x) \\
& -\left(\frac{1}{4} + \frac{1}{2}x - \frac{3}{4}x^2\right)\text{Li}_2(x) + \frac{3}{4}x^2\zeta_2 + \frac{1}{2}x^2\zeta_3 \\
& + \frac{1}{2}(1-x^2)[S_{1,2}(x) - \text{Li}_3(x)] \\
& + \frac{1}{2}(1-x^2)[+\ln(1-x)\text{Li}_2(x)] \tag{244}
\end{aligned}$$

$$\begin{aligned}
\int_0^1 dy \int_0^1 dz \frac{z \ln^2(1-xy)}{1-zy} &= 2S_{1,2}(x) + 2x\text{Li}_3(x) + 2(1-x)\ln(1-x)\text{Li}_2(x) \\
&+ \frac{2}{3}(1-x)\ln^3(1-x) \tag{245}
\end{aligned}$$

$$\begin{aligned}
\int_0^1 dy \int_0^1 dz \frac{yz^2 \ln^2(1-xy)}{1-zy} &= -1 + 2(1-x)\ln(1-x)\text{Li}_2(x) + 2S_{1,2}(x) + 2x\text{Li}_3(x) \\
&+ \left(1 - \frac{1}{x}\right)\ln(1-x) - \frac{1}{2}\left(1 - \frac{1}{x}\right)\ln^2(1-x) \\
&+ \frac{2}{3}(1-x)\ln^3(1-x) \tag{246}
\end{aligned}$$

$$\begin{aligned}
\int_0^1 dy \int_0^1 dz \frac{z \ln(y) \ln(1-xy)}{1-zy} &= x(\zeta_3 - \zeta_2) - (1-2x)\text{Li}_3(x) - \zeta_2(1-x)\ln(1-x) \\
&+ (2-x)S_{1,2}(x) + (1-x)\ln(1-x)\text{Li}_2(x) \\
&+ \frac{1}{2}(1-x)\ln^2(1-x) - x\text{Li}_2(x) \tag{247}
\end{aligned}$$

$$\begin{aligned}
\int_0^1 dy \int_0^1 dz \frac{z \ln(1-y) \ln(1-xy)}{1-zy} &= 2x\zeta_3 + 2(1-x)\ln(1-x)\text{Li}_2(x) + (2x-1)\text{Li}_3(x) \\
&+ (3-2x)S_{1,2}(x) + \frac{1}{3}(1-x)\ln^3(1-x) \tag{248}
\end{aligned}$$

$$\begin{aligned}
\int_0^1 dy \int_0^1 dz \frac{yz^3 \ln(1-y) \ln(1-xy)}{1-zy} &= x\zeta_2 - \frac{2}{3} + x^2\zeta_3 + (1-x^2)\ln(1-x)\text{Li}_2(x) \\
&+ \left(x^2 - \frac{1}{2}\right)\text{Li}_3(x) + \left(\frac{3}{2} - x^2\right)S_{1,2}(x) \\
&+ \frac{x-1}{3x} \left[\ln(1-x) - \text{Li}_2(x) - \frac{1}{2}\ln^2(1-x) \right] \\
&+ \frac{1}{6}(1-x^2)\ln^3(1-x) \tag{249}
\end{aligned}$$

$$\begin{aligned}
\int_0^1 dy \int_0^1 dz \frac{z \ln(1-z) \ln(1-xy)}{1-zy} &= (1-x) \left[2\zeta_2 \ln(1-x) - \frac{1}{2}\ln^2(1-x) - \text{Li}_2(x) \right. \\
&- 2\text{Li}_3(1-x) + \ln(1-x)\text{Li}_2(1-x) \\
&\left. + \frac{1}{6}\ln^3(1-x) \right] + x\zeta_2 + 2\zeta_3 \tag{250}
\end{aligned}$$

$$\begin{aligned}
\int_0^1 dy \int_0^1 dz \frac{z^2 \ln(1-z) \ln(1-xy)}{1-zy} &= \frac{5x}{4} + x^2\zeta_3 + \frac{1}{2}(1-x^2)\ln(1-x)[\zeta_2 + \text{Li}_2(x)] \\
&- \frac{3}{4}(1-x)\ln(1-x) - \frac{1}{4}(3-2x-x^2)\text{Li}_2(x) \\
&+ (1-x^2)S_{1,2}(x) + \frac{1}{2}\left(x + \frac{x^2}{2}\right)\zeta_2
\end{aligned}$$

$$\begin{aligned}
& +\frac{1}{8} (2x-3+x^2) \ln^2(1-x) \\
& +\frac{1}{12} (1-x^2) \ln^3(1-x)
\end{aligned} \tag{251}$$

$$\begin{aligned}
\int_0^1 dy \int_0^1 dz \frac{z^3 \ln(1-z) \ln(1-xy)}{1-zy} = & \frac{5x}{6} + \frac{7x^2}{9} + \left(-\frac{11}{12} + \frac{17}{36}x + \frac{4}{9}x^2 \right) \ln(1-x) \\
& + \left(-\frac{11}{18} + \frac{x}{3} + \frac{x^2}{6} + \frac{x^3}{9} \right) \text{Li}_2(x) + \frac{2}{3}x^3\zeta_3 \\
& + \frac{2}{3} (1-x^3) S_{1,2}(x) + \left(\frac{x}{3} + \frac{x^2}{6} + \frac{x^3}{9} \right) \zeta_2 \\
& + \left(-\frac{11}{36} + \frac{x}{6} + \frac{x^2}{12} + \frac{x^3}{18} \right) \ln^2(1-x) \\
& + \frac{1}{3} (1-x^3) \left\{ \ln(1-x) [\text{Li}_2(x) + \zeta_2] \right. \\
& \left. + \frac{1}{3} (1-x^3) \left\{ +\frac{1}{6} \ln^3(1-x) \right\} \right\}
\end{aligned} \tag{252}$$

$$\begin{aligned}
\int_0^1 dy \int_0^1 dz \frac{yz^2 \ln(1-z) \ln(1-xy)}{1-zy} = & 2x\zeta_3 - \frac{3}{4} + x\zeta_2 + \frac{3}{4} \left(1 - \frac{1}{x} \right) \ln(1-x) \\
& - \frac{1}{2} (1-x) \ln^2(1-x) + \frac{1}{6} (1-x) \ln^3(1-x) \\
& - (1-x) \text{Li}_2(x) + 2(1-x) S_{1,2}(x) \\
& + (1-x) \ln(1-x) [\text{Li}_2(x) + \zeta_2]
\end{aligned} \tag{253}$$

$$\begin{aligned}
\int_0^1 dy \int_0^1 dz \frac{yz^3 \ln(1-z) \ln(1-xy)}{1-zy} = & \frac{5x}{4} - \frac{11}{18} + x^2\zeta_3 + \frac{1}{8} (2x-3+x^2) \ln^2(1-x) \\
& + \frac{1}{12} (1-x^2) \ln^3(1-x) + (1-x^2) S_{1,2}(x) \\
& + \frac{1}{2} (1-x^2) \ln(1-x) [\zeta_2 + \text{Li}_2(x)] \\
& + \left(\frac{3}{4}x - \frac{5}{36} - \frac{11}{18x} \right) \ln(1-x) + \frac{2x+x^2}{4} \zeta_2 \\
& + \frac{1}{4} (-3+2x+x^2) \text{Li}_2(x)
\end{aligned} \tag{254}$$

$$\begin{aligned}
\int_0^1 dy \int_0^1 dz \frac{z \ln(1-yz) \ln(1-xy)}{1-zy} = & x(\zeta_3 + \zeta_2) - (1-x) [\text{Li}_3(x) - S_{1,2}(x) + \text{Li}_2(x)] \\
& + \frac{1}{2} (x-1) \ln^2(1-x) + \frac{1}{6} (1-x) \ln^3(1-x) \\
& + (1-x) \ln(1-x) \text{Li}_2(x)
\end{aligned} \tag{255}$$

$$\begin{aligned}
\int_0^1 dy \int_0^1 dz \frac{yz^2 \ln(1-yz) \ln(1-xy)}{1-zy} = & x\zeta_3 - \frac{5}{4} + \frac{3}{4} \left(1 - \frac{1}{x} \right) \ln(1-x) + \frac{3}{2}x\zeta_2 \\
& + (1-x) \left[\ln(1-x) \text{Li}_2(x) + \frac{1}{6} \ln^3(1-x) \right] \\
& + \left(-1 + \frac{3x}{4} + \frac{1}{4x} \right) \ln^2(1-x) \\
& + \left(\frac{1}{2x} - 2 + \frac{3x}{2} \right) \text{Li}_2(x) \\
& + (1-x) [S_{1,2}(x) - \text{Li}_3(x)]
\end{aligned} \tag{256}$$

$$\begin{aligned}
\int_0^1 dy \int_0^1 dz \frac{z \ln(y) \ln(1-xy)}{(1-zy)^2} &= x(\zeta_2 - \zeta_3) + x \ln(1-x) \text{Li}_2(x) \\
&+ x [S_{1,2}(x) - 2\text{Li}_3(x)] + x \text{Li}_2(x) \\
&- \frac{1}{2}(1-x) \ln^2(1-x) - x \zeta_2 \ln(1-x) \quad (257)
\end{aligned}$$

$$\begin{aligned}
\int_0^1 dy \int_0^1 dz \frac{z^2 \ln(y) \ln(1-xy)}{(1-zy)^2} &= \frac{3}{2}x + x^2 \ln(1-x) [\text{Li}_2(x) - \zeta_2] + x \left(\frac{x}{2} + 1\right) \text{Li}_2(x) \\
&+ x^2 [S_{1,2}(x) - 2\text{Li}_3(x)] + \frac{x-1}{2} \ln(1-x) - x^2 \zeta_3 \\
&- x \zeta_2 \left(1 - \frac{x}{2}\right) - \frac{1}{4} (1-x^2) \ln^2(1-x) \quad (258)
\end{aligned}$$

$$\begin{aligned}
\int_0^1 dy \int_0^1 dz \frac{z^3 \ln(y) \ln(1-xy)}{(1-zy)^2} &= \frac{5x}{8} + \frac{4x^2}{3} - x^3 \zeta_3 + x^3 [S_{1,2}(x) - 2\text{Li}_3(x)] \\
&- \left(\frac{1}{2} - \frac{1}{3}x^2 - \frac{1}{6}x\right) \ln(1-x) + \left(\frac{x^3}{3} - \frac{x}{2} - x^2\right) \zeta_2 \\
&+ \left(x^2 + \frac{1}{3}x^3 + \frac{1}{2}x\right) \text{Li}_2(x) + x^3 \ln(1-x) \text{Li}_2(x) \\
&- \zeta_2 x^3 \ln(1-x) - \frac{1}{6} (1-x^3) \ln^2(1-x) \quad (259)
\end{aligned}$$

$$\begin{aligned}
\int_0^1 dy \int_0^1 dz \frac{yz^2 \ln(y) \ln(1-xy)}{(1-zy)^2} &= -2x(\zeta_3 - \zeta_2) - (1-2x) \ln(1-x) [\text{Li}_2(x) - \zeta_2] \\
&- 2(1-x) S_{1,2}(x) + (1-4x) \text{Li}_3(x) + 2x \text{Li}_2(x) \\
&- (1-x) \ln^2(1-x) \quad (260)
\end{aligned}$$

The following integrals were also required. They are understood in the sense of an analytic continuation and expanding in ε to the order needed in the present calculation.

$$\begin{aligned}
\int_0^1 dy y^{-2-\varepsilon} \ln(1+uy) &= -\frac{u}{\varepsilon} + u - (1+u) \ln(1+u) \\
&+ \varepsilon [u \text{Li}_2(-u) + (1+u) \ln(1+u) - u] \quad (261)
\end{aligned}$$

$$\begin{aligned}
\int_0^1 dy y^{-3-\varepsilon} \ln(1+uy) &= \frac{u^2}{2\varepsilon} - \frac{u^2}{4} - \frac{u}{2} - \frac{1}{2}(1-u^2) \ln(1+u) \\
&+ \varepsilon \left[\frac{u^2}{8} + \frac{3}{4}u - \frac{u^2}{2} \text{Li}_2(-u) + \frac{1}{4}(1-u^2) \ln(1+u) \right] \quad (262)
\end{aligned}$$

$$\begin{aligned}
\int_0^1 dy y^{-4-\varepsilon} \ln(1+uy) &= -\frac{u^3}{3\varepsilon} + \frac{u^2}{3} - \frac{u}{6} + \frac{u^3}{9} - \frac{1}{3}(1+u^3) \ln(1+u) \\
&+ \varepsilon \left[\frac{1}{3}u^3 \text{Li}_2(-u) - \frac{u^3}{27} - \frac{4}{9}u^2 + \frac{5}{36}u \right. \\
&\quad \left. + \frac{1}{9}(1+u^3) \ln(1+u) \right] \quad (263)
\end{aligned}$$

$$\int_0^1 dy y^{-2-\varepsilon} \ln^2(1+uy) = -2u \text{Li}_2(-u) - (1+u) \ln^2(1+u) \quad (264)$$

$$\begin{aligned}
\int_0^1 dy y^{-3-\varepsilon} \ln^2(1+uy) &= -\frac{u^2}{\varepsilon} + \frac{3}{2}u^2 - u(1+u) \ln(1+u) + u^2 \text{Li}_2(-u) \\
&- \frac{1}{2}(1-u^2) \ln^2(1+u) \quad (265)
\end{aligned}$$

$$\begin{aligned} \int_0^1 dy y^{-4-\varepsilon} \ln^2(1+uy) &= \frac{u^3}{\varepsilon} - \frac{1}{3}(1+u)(1-u+u^2) \ln^2(1+u) - \frac{1}{6}(2+7u)u^2 \\ &\quad - \frac{u}{3}(1+u)(1-3u) \ln(1+u) - \frac{2}{3}u^3 \text{Li}_2(-u) \end{aligned} \quad (266)$$

$$\begin{aligned} \int_0^1 dy y^{-3-\varepsilon} \ln(1+u_1y) \ln(1+u_2y) &= -\frac{1}{2\varepsilon}u_1u_2 + \frac{3}{4}u_1u_2 - \frac{1}{4}\ln(1+u_1) \ln(1+u_2) \\ &\quad + \frac{u_1^2}{2} \left[\text{Li}_2(-u_2) - \text{Li}_2\left(\frac{u_1}{u_1-u_2}\right) + \text{Li}_2\left(\frac{u_1(1+u_2)}{u_1-u_2}\right) \right. \\ &\quad \left. + \ln(1+u_2) \ln\left(\frac{u_2(1+u_1)}{u_2-u_1}\right) \right] \\ &\quad - \frac{u_2}{2}(1+u_1) \ln(1+u_1) + \{u_1 \leftrightarrow u_2\} \end{aligned} \quad (267)$$

$$\begin{aligned} \int_0^1 dy y^{-4-\varepsilon} \ln(1+u_1y) \ln(1+u_2y) &= \frac{1}{2\varepsilon}u_1^2u_2 - \frac{1}{6}u_2u_1^2 - \frac{1}{6}\ln(1+u_1) \ln(1+u_2) \\ &\quad - \frac{u_1^2}{3} [u_2 - (1+u_2) \ln(1+u_2)] - \frac{u_1^3}{3} \text{Li}_2(-u_2) \\ &\quad - \frac{u_1}{6} \left[u_2 + (1-u_2^2) \ln(1+u_2) + \frac{u_2^2}{2} \right] \\ &\quad + \frac{u_1^3}{3} \text{Li}_2\left(\frac{u_1}{u_1-u_2}\right) - \frac{u_1^3}{3} \text{Li}_2\left(\frac{u_1(1+u_2)}{u_1-u_2}\right) \\ &\quad - \frac{u_1^3}{3} \ln(1+u_2) \ln\left(\frac{u_2(1+u_1)}{u_2-u_1}\right) + \{u_1 \leftrightarrow u_2\} \end{aligned} \quad (268)$$

$$\begin{aligned} \int_0^1 dy y(1-y) \text{Li}_2\left(-\frac{x}{(1-x)^2y(1-y)}\right) &= -\frac{4}{9}\frac{x}{(1-x)^2} + \frac{5}{9}\ln(1-x) - \frac{1}{3}\ln^2(1-x) \\ &\quad + \frac{1}{3}\ln(x) \ln(1-x) + \frac{1}{9}\frac{x(5x^2-12x+3)}{(1-x)^3} \ln(x) . \end{aligned} \quad (269)$$

B.5 Results for the Feynman integrals

We will now present the results for all of the integrals appearing in equations (106) to (113). We give the results up to $O(\varepsilon^0)$ in x -space. Logarithms, polylogarithms and Nielsen functions appear repeatedly in the expressions, so in order to save space, we use the following shorthand notation:

$$\begin{aligned} L_1 &= \ln(x), & L_2 &= \ln(1-x), & L_3 &= \ln(1+x), & P_1 &= \ln(x), \\ P_2 &= \text{Li}_2(1-x), & P_3 &= \text{Li}_2(-x), & R_1 &= \text{Li}_3(x), & R_2 &= \text{Li}_3(1-x), \\ R_3 &= \text{Li}_3(-x), & R_4 &= \text{S}_{1,2}(x), & R_5 &= \text{S}_{1,2}(1-x), & R_6 &= \text{S}_{1,2}(-x) . \end{aligned}$$

Let us define for $X = A, B, C, E$

$$X_{\nu_1\nu_2\nu_3\nu_4\nu_5}^{a,b} = \int_0^1 dx x^n (m^2)^{4-\nu_{12345}+\varepsilon} (\Delta \cdot p)^{a+b} \tilde{X}_{\nu_1\nu_2\nu_3\nu_4\nu_5}^{a,b} \quad (270)$$

$$\text{and} \quad F_{\nu_1\nu_2\nu_3\nu_4\nu_5}^{a,b} = \int_0^1 dx (x^n - 1) (m^2)^{4-\nu_{12345}+\varepsilon} (\Delta \cdot p)^{a+b} \tilde{F}_{\nu_1\nu_2\nu_3\nu_4\nu_5}^{a,b} . \quad (271)$$

The A -type integrals are

$$\tilde{A}_{01111}^{0,n} = \frac{2}{\varepsilon^2} L_1 - \frac{1}{\varepsilon} \left[4P_1 - 4\zeta_2 - \frac{1}{2} L_1^2 \right] + \frac{1}{2} \zeta_2 L_1 - 6\zeta_3 - \frac{1}{12} L_1^3 - 2R_1 + 8R_4 \quad (272)$$

$$\begin{aligned} \tilde{A}_{01111}^{1,n} &= \frac{2}{\varepsilon^2} [1 - x + (1+x)L_1] - \frac{1}{\varepsilon} \left[(1+x) \left(4P_1 - L_1 - \frac{1}{2} L_1^2 - 4\zeta_2 \right) \right. \\ &\quad \left. + (1-x)(2 - 4L_2) \right] + \frac{1}{4} (3x-1)L_1^2 + (1+x) \left[\frac{1}{2} \zeta_2 L_1 - 6\zeta_3 - 2P_1 \right. \\ &\quad \left. + 2\zeta_2 - L_1 - 2R_1 + 8R_4 - \frac{1}{12} L_1^3 \right] + (1-x) \left[4L_2^2 - 4L_2 - \frac{3}{2} \zeta_2 + 2 \right] \end{aligned} \quad (273)$$

$$\tilde{A}_{02111}^{0,n} = P_2 + \frac{1}{2} L_1^2 + \zeta_2 \quad (274)$$

$$\begin{aligned} \tilde{A}_{10111}^{0,n} &= -\frac{4}{\varepsilon^2} (1-x) - \frac{8}{\varepsilon} (1-x) [L_2 - 1] + (1-x) (3\zeta_2 + 16L_2 - 8L_2^2 - 16) \\ &\quad + (-1)^n \left[-\frac{4}{\varepsilon^2} (1-x) + \frac{8}{\varepsilon} (1-x + xL_1) + 8(1+x) (P_3 + L_1 L_3) \right. \\ &\quad \left. + (3+5x)\zeta_2 + 4xL_1^2 - 16xL_1 - 16(1-x) \right] \end{aligned} \quad (275)$$

$$\begin{aligned} \tilde{A}_{11011}^{0,n} &= (-1)^n \left[\frac{2}{\varepsilon^2} L_1 + \frac{5}{2\varepsilon} L_1^2 + 4R_1 + 8R_3 + 2\zeta_3 - 4L_1 P_3 - 2L_1 P_1 \right. \\ &\quad \left. + \frac{7}{12} L_1^3 + \frac{1}{2} \zeta_2 L_1 \right] \end{aligned} \quad (276)$$

$$\begin{aligned} \tilde{A}_{11011}^{1,n} &= (-1)^n \left\{ -\frac{2}{\varepsilon^2} [xL_1 + 1 - x] - \frac{1}{\varepsilon} \left[2x - 2 + \frac{5}{2} xL_1^2 + (1-3x)L_1 \right] \right. \\ &\quad \left. + \frac{1}{4} (1+9x)L_1^2 + 4xL_1 P_3 - \frac{1}{2} (1-x)\zeta_2 + 2xL_1 P_1 - \frac{1}{2} x\zeta_2 L_1 \right. \\ &\quad \left. - \frac{7}{12} xL_1^3 - 2\zeta_3 x - 4xR_1 - 8xR_3 + (1-3x)L_1 + 2x - 2 \right\} \end{aligned} \quad (277)$$

$$\tilde{A}_{12011}^{0,n} = (-1)^n \left[\frac{3}{2} L_1^2 - P_1 - 2P_3 + 2\frac{L_1}{1+x} - L_1 L_2 - 2L_1 L_3 \right] \quad (278)$$

$$\tilde{A}_{11101}^{0,n} = \tilde{A}_{11011}^{0,n} \quad (279)$$

$$\begin{aligned} \tilde{A}_{11101}^{1,n} &= (-1)^n \left\{ \frac{2}{\varepsilon^2} [1 - x + L_1] - \frac{1}{\varepsilon} \left[2 - 2x - \frac{5}{2} L_1^2 + (3x-1)L_1 \right] + \frac{1}{2} \zeta_2 L_1 \right. \\ &\quad \left. - \frac{1}{4} (1+9x)L_1^2 + \frac{1}{2} (1-x)\zeta_2 + 2\zeta_3 + \frac{7}{12} L_1^3 - 4L_1 P_3 - 2L_1 P_1 \right. \\ &\quad \left. + 4R_1 + 8R_3 + (3x-1)L_1 + 2 - 2x \right\} \end{aligned} \quad (280)$$

$$\tilde{A}_{12101}^{0,n} = \tilde{A}_{12011}^{0,n} \quad (281)$$

$$\tilde{A}_{11110}^{0,n} = \frac{2}{\varepsilon^2} L_1 - \frac{1}{\varepsilon} \left[4P_1 - 4\zeta_2 - \frac{1}{2} L_1^2 \right] - 6\zeta_3 + \frac{1}{2} \zeta_2 L_1 - \frac{1}{12} L_1^3 - 2R_1 + 8R_4 \quad (282)$$

$$\begin{aligned} \tilde{A}_{11110}^{1,n} &= -\frac{2}{\varepsilon^2} (1-x) - \frac{1}{\varepsilon} [(1+x)L_1 + 4(1-x)L_2 - 2 + 2x] + \frac{1}{4} (1-3x)L_1^2 \\ &\quad - (1-x) (4L_2^2 - 4L_2 + 2) + (1+x) (L_1 + 2P_1) - \frac{1}{2} (1+7x)\zeta_2 \end{aligned} \quad (283)$$

$$\tilde{A}_{12110}^{0,n} = P_2 + \frac{1}{2}L_1^2 + \zeta_2 \quad (284)$$

$$\begin{aligned} \tilde{A}_{22110}^{0,n} = & -\left(\frac{1}{2\varepsilon^2} - \frac{1}{2\varepsilon} - \frac{1}{2} + \frac{5}{8}\zeta_2\right) \delta(1-x) - \left(\frac{1}{\varepsilon} - 1\right) \mathcal{D}_0(x) \\ & - 2\mathcal{D}_1(x) + \frac{L_1}{1-x} \end{aligned} \quad (285)$$

$$\tilde{A}_{21110}^{0,n} = -\frac{2}{\varepsilon^2} - \frac{2}{\varepsilon} [2L_2 - L_1] + 2P_2 - 4L_2^2 + 4L_1L_2 - \frac{1}{2}\zeta_2 \quad (286)$$

$$\tilde{A}_{21110}^{1,n} = -\frac{2}{\varepsilon^2} - \frac{4}{\varepsilon}L_2 + \left(\frac{1}{2} - x\right) \zeta_2 - 4L_2^2 - (1+x)P_2 - \frac{1}{2}xL_1^2 \quad (287)$$

$$\begin{aligned} \tilde{A}_{01111}^{n,0} = & \frac{2}{\varepsilon^2}L_2 - \frac{1}{\varepsilon} \left[4P_2 - 4\zeta_2 - \frac{1}{2}L_2^2 \right] + \frac{5}{2}\zeta_2L_2 - 2\zeta_3 - L_2P_2 + \frac{1}{12}L_2^3 \\ & - 2R_2 + 4R_5 \end{aligned} \quad (288)$$

$$\begin{aligned} \tilde{A}_{01111}^{n,1} = & -\frac{2}{\varepsilon^2} [x + (1-x)L_2] - \frac{1}{\varepsilon} \left[(2-x)L_2 + (1-x) \left(\frac{1}{2}L_2^2 + 4\zeta_2 - 4P_2 \right) \right. \\ & \left. + 4xL_1 - 2x \right] - 4L_1L_2 - \frac{1}{4}(2-x)L_2^2 - 2xL_1^2 + \frac{x}{2}\zeta_2 + (3x-2)L_2 \\ & - 2x - (1-x) \left[L_2P_1 + \frac{3}{2}\zeta_2L_2 - 2\zeta_3 + L_1L_2^2 + \frac{1}{12}L_2^3 + 4R_5 - 2R_2 \right] \\ & + (x-4)P_1 \end{aligned} \quad (289)$$

$$\tilde{A}_{10111}^{n,0} = -\frac{4}{\varepsilon^2} - \frac{4}{\varepsilon} [L_2 - 1] - 4 - 2L_2^2 + 4L_2 - \zeta_2 \quad (290)$$

$$\tilde{A}_{20111}^{n,0} = -\left(\frac{2}{\varepsilon^2} - \frac{2}{\varepsilon} + 2 + \frac{1}{2}\zeta_2\right) \delta(1-x) - \frac{2}{\varepsilon} \mathcal{D}_0(x) + 2\mathcal{D}_0(x) - 2\mathcal{D}_1(x) \quad (291)$$

$$\tilde{A}_{11011}^{n,0} = \frac{2}{\varepsilon^2}L_1 + \frac{5}{2\varepsilon}L_1^2 + 4R_1 + 8R_3 + 2\zeta_3 - 4L_1P_3 - 2L_1P_1 + \frac{7}{12}L_1^3 + \frac{1}{2}\zeta_2L_1 \quad (292)$$

$$\begin{aligned} \tilde{A}_{11011}^{n,1} = & \frac{2}{\varepsilon^2} [xL_1 + 1 - x] - \frac{1}{\varepsilon} \left[(3x-1)L_1 - \frac{5}{2}xL_1^2 + 2 - 2x \right] + \frac{1}{2}(1-x)\zeta_2 \\ & + \frac{x}{2}\zeta_2L_1 - 2xL_1P_1 + 2\zeta_3x + 8xR_3 + (3x-1)L_1 + 4xR_1 + \frac{7}{12}xL_1^3 \\ & - \frac{1}{4}(1+9x)L_1^2 - 4xL_1P_3 + 2 - 2x \end{aligned} \quad (293)$$

$$\tilde{A}_{21011}^{n,0} = P_2 - 2P_3 - \zeta_2 + \frac{3}{2}L_1^2 + 2\frac{L_1}{1+x} - 2L_1L_3 \quad (294)$$

$$\tilde{A}_{11101}^{n,0} = -\frac{2}{\varepsilon^2} - \frac{1}{\varepsilon} [4L_2 - L_1 - 2] - \frac{1}{2}\zeta_2 - \frac{1}{4}L_1^2 - \frac{1+3x}{1-x}L_1 - 2L_2^2 - 2 \quad (295)$$

$$\begin{aligned} \tilde{A}_{11101}^{n,1} = & \frac{1-x}{\varepsilon^2} - \frac{1+x}{\varepsilon} \left[\frac{1}{2}L_1 - 2L_2 + 1 \right] + (1-x) \left[L_2^2 + \frac{1}{8}L_1^2 + \frac{1}{4}\zeta_2 + 1 \right] \\ & + \frac{1}{2}(1+3x)L_1 \end{aligned} \quad (296)$$

$$\tilde{A}_{21101}^{n,0} = -\left(\frac{2}{\varepsilon^2} + \frac{\zeta_2}{2}\right) \delta(1-x) - \frac{2}{\varepsilon} \mathcal{D}_0(x) - 2\mathcal{D}_1(x) + 1 + \frac{1+x}{1-x} \left(1 + \frac{L_1}{1-x} \right) \quad (297)$$

$$\tilde{A}_{11110}^{n,0} = -\frac{2}{\varepsilon^2} - \frac{1}{\varepsilon} (3L_2 - 2) - \frac{7}{4}L_2^2 + \frac{1}{2}\zeta_2 + P_1 + 3L_2 - 2 \quad (298)$$

$$\tilde{A}_{11110}^{n,1} = -\frac{x}{\varepsilon^2} - \frac{x}{\varepsilon} \left(\frac{3}{2}L_2 - 1 \right) + \frac{x}{4}\zeta_2 - \frac{7}{8}xL_2^2 + \frac{1}{2}(3x+1)L_2 + \frac{x}{2}P_1 - x \quad (299)$$

$$\begin{aligned} \tilde{A}_{21110}^{n,0} &= -\left(\frac{2}{\varepsilon^2} - \frac{2}{\varepsilon} + 2 + \frac{1}{2}\zeta_2 \right) \delta(1-x) - \frac{2}{\varepsilon}\mathcal{D}_0(x) + 2\mathcal{D}_0(x) - 2\mathcal{D}_1(x) \\ &\quad - P_1 - \frac{1}{2}L_2^2 - \zeta_2 \end{aligned} \quad (300)$$

$$\begin{aligned} \tilde{A}_{31110}^{n,0} &= \left(\frac{1}{2\varepsilon^2} - \frac{1}{\varepsilon} - \frac{1}{2} + 1 - \frac{3}{8}\zeta_2 \right) \delta(1-x) - \mathcal{D}_0(x) - \mathcal{D}_1(x) \\ &\quad - (1-x)^{-2+\varepsilon} \left[\frac{3}{2} - \frac{1}{\varepsilon} - \left(\frac{3}{2} + \frac{1}{4}\zeta_2 \right) \varepsilon \right] + \zeta_2 + \frac{1}{2}L_2^2 + P_1 \end{aligned} \quad (301)$$

$$\begin{aligned} \tilde{A}_{21110}^{n,1} &= -\left(\frac{1}{\varepsilon^2} - \frac{3}{2\varepsilon} + \frac{7}{4} + \frac{1}{4}\zeta_2 \right) \delta(1-x) - \frac{1}{\varepsilon}(\mathcal{D}_0(x) - 1) + \frac{3}{2}\mathcal{D}_0(x) \\ &\quad - \mathcal{D}_1(x) - \frac{3}{2} - x\zeta_2 - \frac{x}{2}L_2^2 + \frac{1}{2}L_2 - xP_1 \end{aligned} \quad (302)$$

$$\begin{aligned} \tilde{A}_{12111}^{0,n} &= -\frac{1}{3\varepsilon}\delta(1-x) - \frac{2}{3}\mathcal{D}_0(x) - \frac{2}{3}P_2 - \frac{1}{3}L_1^2 - \frac{2}{3}\zeta_2 - \frac{2}{3(1-x)}L_1 \\ &\quad + \frac{1}{3}(1-x)^{-3+2\varepsilon}L_1^2 + \frac{2}{3}(1-x)^{-2+2\varepsilon}L_1 - \left(\frac{1}{2} - \zeta_2 \right) \delta(1-x) \\ &\quad - (-1)^n \left[\frac{4}{3}P_3 - \frac{2}{3} \left(1 - \frac{2}{(1+x)^3} \right) P_2 + \frac{2}{3}\zeta_2 - \left(1 - \frac{1}{(1+x)^3} \right) L_1^2 \right. \\ &\quad + \frac{4}{3}L_1L_3 + \frac{4}{3(1+x)^2} + \frac{2}{3(1+x)} \left(\frac{2}{(1+x)^2} - \frac{1}{1+x} - 1 \right) L_1 \\ &\quad \left. - \frac{2}{3(1+x)} \right] \end{aligned} \quad (303)$$

$$\begin{aligned} \tilde{A}_{21111}^{n,0} &= \left(-\frac{1}{2\varepsilon^2} + \frac{3}{8}\zeta_2 + \frac{1}{2} \right) \delta(1-x) + \mathcal{D}_1(x) + \left(\frac{1}{(1-x)^2} - \frac{3}{2} \right) \zeta_2 - \frac{1}{4}L_2^2 \\ &\quad + \left(\frac{2}{(1-x)^2} - 1 \right) (L_1L_3 + P_3) + \left(1 - \frac{1}{2(1-x)^2} \right) P_2 - \frac{L_2}{2(1-x)} \\ &\quad + \frac{1}{(1-x)^2} \left[\frac{3}{4}x(x-2)L_1^2 + \frac{x(x-3)}{2(1+x)}L_1 + \frac{x}{2}(x-2)L_1L_2 \right] \end{aligned} \quad (304)$$

The integrals $A_{11111}^{0,n}$, $A_{11111}^{n,0}$ and $A_{11111}^{n,1}$ are finite and appear multiplied by a factor of $D-4 = \varepsilon$ in the expressions of diagrams 5a and 5c. Therefore they only contribute to those diagrams starting at $O(\varepsilon)$. For this reason we have chosen not to present them here, although they are needed in Eq. (124).

The B -type integrals are given by

$$\begin{aligned} \tilde{B}_{12011}^{0,n+1} &= (-1)^n \left\{ -\frac{1-x}{\varepsilon^2} - \frac{1}{\varepsilon} \left[\frac{1}{2}(5-x)L_1 + 2 - 2x \right] + 2x - 2 - \frac{1}{4}(5-x)\zeta_2 \right. \\ &\quad \left. + P_2 - \frac{1}{8}(7+x)L_1^2 + 2\frac{x^2-1-x}{1+x}L_1 - 2L_1L_3 - 2P_3 \right\} \end{aligned} \quad (305)$$

$$\tilde{B}_{12101}^{0,n+1} = \tilde{B}_{12011}^{0,n+1} \quad (306)$$

$$\begin{aligned}\tilde{B}_{21011}^{n+1,0} &= \frac{x-1}{\varepsilon^2} - \frac{1}{2\varepsilon} [(5-x)L_1 + 5(1-x)] + \frac{x-5}{4}\zeta_2 - \frac{7+x}{8}L_1^2 \\ &\quad - \frac{1}{4}(7+11x)\frac{1-x}{1+x}L_1 + P_2 - 2P_3 - 2L_1L_3 - \frac{3}{2}(1-x)\end{aligned}\quad (307)$$

$$\begin{aligned}\tilde{B}_{21101}^{n+1,0} &= -\frac{x}{2\varepsilon^2} - \frac{1}{4\varepsilon} (4xL_2 - xL_1 + 1 - 3x) - \frac{x}{16}L_1^2 - \frac{x}{8}\zeta_2 \\ &\quad + \frac{1-9x^2}{8(1-x)}L_1 - \frac{x}{2}L_2^2 + \frac{1}{4}(1-3x)\end{aligned}\quad (308)$$

$$\begin{aligned}\tilde{B}_{12110}^{0,n+1} &= -\frac{1-x}{\varepsilon^2} - \frac{1}{\varepsilon} \left[\frac{1}{2} (3x+1)L_1 + 2(1-x)L_2 \right] - \frac{1}{4}(1-x)\zeta_2 \\ &\quad + \frac{1}{8}(1-x)L_1^2 - 3xL_1L_2 - 2xP_2 - 2(1-x)L_2^2 + P_1\end{aligned}\quad (309)$$

$$\begin{aligned}\tilde{B}_{21110}^{n+1,0} &= -\left(\frac{1}{\varepsilon^2} - \frac{3}{2\varepsilon} + \frac{7}{4} + \frac{1}{4}\zeta_2 \right) \delta(1-x) - \frac{x}{2\varepsilon^2} - \frac{1}{\varepsilon} \left(\mathcal{D}_0(x) + \frac{3}{4}xL_2 - \frac{x}{2} - 1 \right) \\ &\quad + \frac{3}{2}\mathcal{D}_0(x) - \mathcal{D}_1(x) - \frac{3}{2} - \frac{7}{8}x\zeta_2 - \frac{15}{16}xL_2^2 + \frac{3}{4}(1+x)L_2 - \frac{3}{4}xP_1 - \frac{x}{2}\end{aligned}\quad (310)$$

The E -type integrals are

$$\begin{aligned}\tilde{E}_{01111}^{0,0} &= \frac{4}{\varepsilon^2} [L_2 - L_1] - \frac{1}{\varepsilon} \left[3L_1^2 - 3L_2^2 - 2P_1 + 2P_2 \right] + \zeta_2L_1 + \zeta_2L_2 - \frac{1}{2}L_1^3 \\ &\quad + 5R_2 - 2L_2L_1^2 - 2L_1P_2 - \frac{5}{2}L_1L_2^2 + \frac{11}{6}L_2^3 - 3R_5 - 7L_2P_2 + 3\zeta_3\end{aligned}\quad (311)$$

$$\begin{aligned}\tilde{E}_{01111}^{1,0} &= \frac{2}{\varepsilon^2} [2xL_2 - (1+2x)L_1] - \frac{1}{\varepsilon} \left[-4(1+x)P_1 + \left(\frac{1}{2} + 3x \right) L_1^2 - 3xL_2^2 \right. \\ &\quad \left. - 2xL_1L_2 + (4+2x)\zeta_2 \right] + (3x+4)L_1L_2^2 - \frac{3}{2}xL_1^2L_2 + (3x-2)\zeta_3 \\ &\quad - 3xR_4 - xL_1P_1 - \left(x + \frac{1}{2} \right) \zeta_2L_1 + 2(x+4)R_2 + (2+3x)R_1 \\ &\quad + 4(x+2)L_2P_1 + \frac{1}{12}(1-6x)L_1^3 + \frac{11}{6}xL_2^3 - (8+3x)\zeta_2L_2\end{aligned}\quad (312)$$

$$\begin{aligned}\tilde{E}_{01111}^{0,1} &= -\frac{2}{\varepsilon^2} [(1-2x)L_2 + 2xL_1] - \frac{1}{\varepsilon} \left[4(1-x)P_1 + (4-2x)L_1L_2 + 3xL_1^2 \right. \\ &\quad \left. + \left(\frac{1}{2} - 3x \right) L_2^2 + 2x\zeta_2 \right] - \left(\frac{3}{2}x + 2 \right) L_1^2L_2 - (x+4)L_1P_1 - x\zeta_2L_1 \\ &\quad + \left(\frac{9}{2}x - 1 \right) L_1L_2^2 + (7x-1)L_2P_1 - \left(\frac{3}{2} + 6x \right) \zeta_2L_2 - \frac{x}{2}L_1^3 - 2\zeta_3 \\ &\quad - \frac{1-22x}{12}L_2^3 + (3x+4)R_1 + (5x+2)R_2\end{aligned}\quad (313)$$

$$\begin{aligned}\tilde{E}_{01111}^{1,1} &= \frac{2x}{\varepsilon^2} [(2x-1)L_2 - (2x+1)L_1] - \frac{1}{\varepsilon} \left[2x(x+2)\zeta_2 + \frac{x}{2}(6x+1)L_1^2 \right. \\ &\quad \left. + 2x(2-x)L_1L_2 - 4x^2P_1 - \frac{x}{2}(6x-1)L_2^2 \right] + \frac{x}{2}(3+2x)\zeta_2L_1 \\ &\quad + x(2+3x)\zeta_3 - \frac{x}{2}(5-2x)\zeta_2L_2 - \frac{x}{12}(1-22x)L_2^3 - x(1+2x)L_1^2L_2\end{aligned}$$

$$\begin{aligned}
& -\frac{x}{2}(8+5x)L_1L_2^2 - x(1+x)[2L_1+7L_2]P_2 + \frac{x}{12}(1-6x)L_1^3 \\
& + 5x(2+x)R_2 - 3x(2+x)R_5
\end{aligned} \tag{314}$$

$$\begin{aligned}
\tilde{E}_{01111}^{0,2} = & \frac{2}{\varepsilon^2} [x - 2x^2L_1 + (2x^2 - 2x + 1)L_2] - \frac{1}{\varepsilon} \left[2x + 4xL_1L_2 + 3x^2L_1^2 \right. \\
& - \left(3x^2 - x + \frac{1}{2} \right) L_2^2 + 2x(2-x)P_1 + (x-2)L_2 + (2x^2 - 4x + 4)P_2 \\
& - 4xL_1 - 4\zeta_2(1-x) \left. \right] - \frac{x}{2}\zeta_2 - \left(2x^2 + 3x - \frac{3}{2} \right) \zeta_2L_2 - 3x^2\zeta_2L_1 + 2x \\
& + \frac{x^2}{2}L_1^2L_2 + \left(1 - 2x + \frac{1}{2}x^2 \right) L_1L_2^2 + (3x^2 - 2x + 1)P_1L_2 + x^2P_1L_1 \\
& + 4L_1L_2 - 4x^2L_2P_2 + 2x^2P_2L_1 + 2xL_1^2 + \frac{2-x}{4}L_2^2 - \frac{x^2}{2}L_1^3 \\
& + \frac{1}{12}(22x^2 - 2x + 1)L_2^3 + (4-8x)R_5 + 3x^2R_1 + (5x^2 + 4x - 2)R_2 \\
& + (4-x)P_1 + (2-3x)L_2 + (4x-2)\zeta_3
\end{aligned} \tag{315}$$

$$\begin{aligned}
\tilde{E}_{01111}^{2,0} = & \frac{2}{\varepsilon^2} [2x^2L_2 - (1+2x+2x^2)L_1 + x-1] - \frac{1}{\varepsilon} \left[2x - 2 - 2x^2L_1L_2 \right. \\
& - 3x^2L_2^2 - 4(1+x)^2P_1 + \left(\frac{1}{2} + x + 3x^2 \right) L_1^2 + 4(1-x)L_2 + (1+x)L_1 \\
& + (4+8x+2x^2)\zeta_2 \left. \right] - \frac{1+7x}{2}\zeta_2 - x^2\zeta_2L_2 - \left(\frac{1}{2} + x + x^2 \right) \zeta_2L_1 \\
& + 2x - 2 - \frac{3}{2}x^2L_1^2L_2 + 2x^2L_1L_2^2 + 2x^2L_2P_1 - x^2L_1P_1 + \frac{1-3x}{4}L_1^2 \\
& - 4(1-x)L_2^2 + \left(\frac{1}{12} + \frac{x}{6} - \frac{x^2}{2} \right) L_1^3 + \frac{11}{6}x^2L_2^3 - (8+16x+5x^2)R_4 \\
& + (3x^2+4x+2)R_1 + (1+x)(L_1+2P_1) + 4(1-x)L_2 \\
& + (5x^2+12x+6)\zeta_3
\end{aligned} \tag{316}$$

$$\begin{aligned}
\tilde{E}_{10111}^{0,0} = & \frac{4}{\varepsilon^2}L_2 - \frac{1}{\varepsilon} (4\zeta_2 - 6L_2^2) + 8\zeta_3 + \frac{14}{3}L_2^3 - 7\zeta_2L_2 \\
& + (-1)^n \left[-\frac{4}{\varepsilon^2}L_1 - \frac{2}{\varepsilon} (4P_3 + 4L_1L_3 + L_1^2 + 2\zeta_2) - \frac{2}{3}L_1^3 - 16R_6 \right. \\
& \left. - 16L_3P_3 - 8L_1L_3^2 - 8\zeta_2L_3 + 8\zeta_3 + 8R_3 - 4L_1^2L_3 - 8L_1P_3 - \zeta_2L_1 \right]
\end{aligned} \tag{317}$$

$$\begin{aligned}
\tilde{E}_{10111}^{1,0} = & \frac{4}{\varepsilon^2} (xL_2 + 1 - x) - \frac{1}{\varepsilon} [x(4\zeta_2 - 6L_2^2) + 8(1-x)(1-L_2)] + 8\zeta_3x \\
& + (1-x)(8L_2^2 - 3\zeta_2 - 16L_2 + 16) + \frac{14}{3}xL_2^3 - 7x\zeta_2L_2 \\
& + (-1)^n \left[\frac{4}{\varepsilon^2} (xL_1 + 1 - x) - \frac{2}{\varepsilon} (4xL_1 - 4xL_1L_3 - 4xP_3 - 2x\zeta_2 \right. \\
& - xL_1^2 - 4x + 4) - (3+5x)\zeta_2 + \zeta_2xL_1 + 8x\zeta_2L_3 + 16xL_1 \\
& - 8(1+x)L_1L_3 + 8xL_1L_3^2 + 4xL_3L_1^2 + 16xL_3P_3 + 8xL_1P_3 - 4xL_1^2 \\
& \left. + \frac{2}{3}xL_1^3 - 8(1+x)P_3 - 8xR_3 + 16xR_6 - 8\zeta_3x + 16 - 16x \right]
\end{aligned} \tag{318}$$

$$\begin{aligned}
\tilde{E}_{10111}^{0,1} = & \frac{4}{\varepsilon^2} (xL_2 + 1) - \frac{2}{\varepsilon} (2x\zeta_2 - 2L_2 - 3xL_2^2 + 2) + \zeta_2 - 7x\zeta_2L_2 + 8\zeta_3x + 4 \\
& + \frac{14}{3}xL_2^3 + 2L_2^2 - 4L_2 + (-1)^n x \left[\frac{4}{\varepsilon^2}L_1 + \frac{2}{\varepsilon} (2\zeta_2 + L_1^2 + 4L_1L_3 + 4P_3) \right. \\
& + \zeta_2 (L_1 + 8L_3) - 8\zeta_3 + \frac{2}{3}L_1^3 + 4L_1^2L_3 + 8L_1 (P_3 + L_3^2) + 16L_3P_3 \\
& \left. - 8R_3 + 16R_6 \right]
\end{aligned} \tag{319}$$

$$\begin{aligned}
\tilde{E}_{11011}^{1,0} = & -\frac{2}{\varepsilon^2}(1-x) - \frac{2}{\varepsilon} [(2-x)L_1 + 1-x] - 2 + 2x + (2x-4)L_1 - \frac{x+4}{2}L_1^2 \\
& + (1-x) \left(P_2 - 2P_3 - \frac{3}{2}\zeta_2 - 2L_1L_3 \right) + (-1)^n \left[-\frac{2}{\varepsilon^2} (L_1 + 1-x) \right. \\
& - \frac{1}{\varepsilon} \left(\frac{5}{2}L_1^2 + (4-2x)L_1 - 2x + 2 \right) - \frac{1}{2}\zeta_2L_1 - 2\zeta_3 - \frac{7}{12}L_1^3 - \frac{x+4}{2}L_1^2 \\
& + (2x-4)L_1 + 2L_1P_1 - (1-x) \left(2L_1L_3 + L_2L_1 + 2P_3 + P_1 + \frac{1}{2}\zeta_2 \right) \\
& \left. + 4L_1P_3 - 8R_3 - 4R_1 + 2x - 2 \right]
\end{aligned} \tag{320}$$

$$\tilde{E}_{11011}^{0,1} = (-1)^n \tilde{E}_{11011}^{1,0} \tag{321}$$

$$\begin{aligned}
\tilde{E}_{11011}^{1,1} = & (1 - (-1)^n) x \left[-\frac{2}{\varepsilon^2} (L_1 + 1-x) - \frac{1}{\varepsilon} \left(\frac{5}{2}L_1^2 + (4-2x)L_1 - 2x + 2 \right) \right. \\
& - \frac{1}{2}\zeta_2L_1 - 2\zeta_3 - \frac{7}{12}L_1^3 + (2x-4)L_1 + 2L_1P_1 + 4L_1P_3 - 8R_3 - 4R_1 \\
& \left. - \frac{x+4}{2}L_1^2 - (1-x) \left(\frac{1}{2}\zeta_2 + 2L_1L_3 + L_1L_2 + 2P_3 + P_1 + 2 \right) \right]
\end{aligned} \tag{322}$$

$$\begin{aligned}
\tilde{E}_{11011}^{2,0} = & x \left[-\frac{2}{\varepsilon^2}(1-x) - \frac{2}{\varepsilon} [(2-x)L_1 + 1-x] + 2(x-2)L_1 - \frac{x+4}{2}L_1^2 \right. \\
& + (1-x) \left(P_2 - 2P_3 - \frac{3}{2}\zeta_2 - 2L_1L_3 - 2 \right) \\
& + (-1)^n \left[\frac{2}{\varepsilon^2} (2xL_1 + 1-x^2) + \frac{1}{\varepsilon} \left(5xL_1^2 + (1+2x)(1-x)L_1 \right. \right. \\
& \left. \left. - 2(1-x)^2 \right) - 5x\zeta_2L_1 + 2(1-x^2) + 2xR_5 + 10xR_1 + 16xR_3 \right. \\
& + (1-x) \left(\frac{1}{2}(1+3x)\zeta_2 - xP_2 + 2xP_3 + 2xL_1L_3 - \frac{1}{4}(1+2x)L_1^2 \right) \\
& \left. \left. + 2\zeta_3x - (1+2x^2-7x)L_1 + xL_1 (6P_2 - 8P_3) + 5xL_1^2L_2 + \frac{7}{6}xL_1^3 \right] \right]
\end{aligned} \tag{323}$$

$$\tilde{E}_{11011}^{0,2} = -(-1)^n \tilde{E}_{11011}^{2,0} \tag{324}$$

$$\begin{aligned}
\tilde{E}_{11101}^{0,0} = & \frac{1}{1-x} \left[\frac{2}{\varepsilon^2}L_1 - \frac{1}{\varepsilon} \left(\frac{1}{2}L_1^2 - 4L_1L_2 - 2P_2 \right) + \frac{1}{2}\zeta_2L_1 + \frac{1}{12}L_1^3 + 2L_1L_2^2 \right. \\
& + 2R_2 - R_5 \left. \right] + \frac{(-1)^n}{1+x} \left[-\frac{4}{\varepsilon^2}L_1 - \frac{1}{\varepsilon} (2\zeta_2 + 4P_3 + 3L_1^2 + 4L_1L_3) \right. \\
& \left. - 2\zeta_2L_3 - \zeta_2L_1 + 3\zeta_3 - 4L_3P_3 - 2L_1P_3 - \frac{1}{2}L_1^3 - 3L_1^2L_3 + 2L_1P_1 \right]
\end{aligned}$$

$$\left. -2L_1L_3^2 - 2R_3 - 4R_1 - 4R_6 \right] \quad (325)$$

$$\begin{aligned} \tilde{E}_{11101}^{1,0} = & \frac{x}{1-x} \left[\frac{2}{\varepsilon^2} L_1 - \frac{1}{\varepsilon} \left(\frac{1}{2} L_1^2 - 4L_1L_2 - 2P_2 \right) + \frac{1}{2} \zeta_2 L_1 + \frac{1}{12} L_1^3 + 2L_1L_2^2 \right. \\ & + 2R_2 - R_5 \left. \right] + \frac{(-1)^n}{1+x} \left[-\frac{2-2x}{\varepsilon^2} L_1 + \frac{1}{\varepsilon} \left(2\zeta_2 x + \frac{1}{2} (x-5) L_1^2 \right. \right. \\ & + 4xL_1L_3 + 4xP_3 \left. \right) - \frac{1-x}{2} \zeta_2 L_1 + 2x\zeta_2 L_3 - (5x+2)\zeta_3 + 2L_1P_1 \\ & - \frac{7+x}{12} L_1^3 + 3xL_1^2L_3 + 2(3x+2)L_1P_3 + 2xL_1L_3^2 + 4xL_3P_3 - 4R_1 \\ & + 4xR_6 - 2(3x+4)R_3 \left. \right] \end{aligned} \quad (326)$$

$$\begin{aligned} \tilde{E}_{11101}^{0,1} = & \frac{2}{\varepsilon^2} \left(1 + \frac{x}{1-x} L_1 \right) - \frac{1}{\varepsilon} \left[\frac{x}{1-x} \left(\frac{1}{2} L_1^2 - 2P_2 - 4L_1L_2 \right) + 2 - 4L_2 + L_1 \right] \\ & + \frac{\zeta_2}{2} + \frac{L_1^2}{4} + 2L_2^2 + 2 + \frac{x}{1-x} \left(\frac{1}{2} \zeta_2 L_1 + 2L_1L_2^2 + \frac{1}{12} L_1^3 - R_5 + 2R_2 \right) \\ & + \frac{1+3x}{1-x} L_1 + (-1)^n \frac{x}{1+x} \left[\frac{4}{\varepsilon^2} L_1 + \frac{1}{\varepsilon} (2\zeta_2 + 3L_1^2 + 4P_3 + 4L_1L_3) \right. \\ & + \zeta_2 L_1 + 2\zeta_2 L_3 + \frac{1}{2} L_1^3 + 3L_1^2L_3 + 2L_1P_3 + 2L_1L_3^2 - 2L_1P_1 + 4L_3P_3 \\ & + 2R_3 + 4R_1 + 4R_6 - 3\zeta_3 \left. \right] \end{aligned} \quad (327)$$

$$\begin{aligned} \tilde{E}_{11101}^{1,1} = & \frac{2}{\varepsilon^2} \left(x + \frac{x^2}{1-x} L_1 \right) - \frac{x}{\varepsilon} \left[L_1 - 4L_2 + 2 + \frac{x}{2(1-x)} \left(L_1^2 - 8L_1L_2 - 4P_2 \right) \right] \\ & + \frac{x^2}{1-x} \left[\frac{1}{2} \zeta_2 L_1 + \frac{1}{12} L_1^3 + 2L_1L_2^2 + 2R_2 - R_5 \right] + \frac{x}{2} \zeta_2 + \frac{x}{4} L_1^2 + 2xL_2^2 \\ & + x \frac{1+3x}{1-x} L_1 + 2x + \frac{(-1)^n}{1+x} \left[\frac{2}{\varepsilon^2} x(1-x)L_1 - \frac{1}{\varepsilon} \left(2x^2\zeta_2 + 4x^2P_3 \right. \right. \\ & + 4x^2L_1L_3 - \frac{5-x}{2} xL_1^2 \left. \right) + \frac{x}{2} (1-x)\zeta_2 L_1 + (2+5x)x\zeta_3 + \frac{x}{12} (7+x)L_1^3 \\ & - x^2 (2\zeta_2 L_3 + 3L_1^2L_3 + 2L_1L_3^2 + 4L_3P_3 + 4R_6) - 2xL_1P_1 + 4xR_1 \\ & + 2x(4+3x)R_3 - 2x(2+3x)L_1P_3 \left. \right] \end{aligned} \quad (328)$$

$$\begin{aligned} \tilde{E}_{11101}^{2,0} = & \frac{x^2}{1-x} \left[\frac{2}{\varepsilon^2} L_1 - \frac{1}{\varepsilon} \left(\frac{1}{2} L_1^2 - 4L_1L_2 - 2P_2 \right) + \frac{1}{2} \zeta_2 L_1 + \frac{1}{12} L_1^3 + 2L_1L_2^2 \right. \\ & + 2R_2 - R_5 \left. \right] + (-1)^n \left[-\frac{2}{\varepsilon^2} \left(1-x + \frac{1+x^2}{1+x} L_1 \right) - \frac{1}{\varepsilon} \left(\frac{5+x^2}{2(1+x)} L_1^2 \right. \right. \\ & + (1-3x)L_1 + \frac{x^2}{1+x} (2\zeta_2 + 4P_3 + 4L_1L_3) - 2(1-x) \left. \right) - \frac{1}{2} (1-x)\zeta_2 \\ & + (1-3x)L_1 + \frac{1+9x}{4} L_1^2 - 2\frac{4-3x^2}{1+x} R_3 - \frac{7-x^2}{12(1+x)} L_1^3 - \frac{2-5x^2}{1+x} \zeta_3 \end{aligned}$$

$$\begin{aligned}
& -\frac{x^2}{1+x} (4R_6 + 2\zeta_2 L_3 + 2L_1 L_3^2 + 4L_3 P_3 + 3L_1^2 L_3) + 2\frac{2-3x^2}{1+x} L_1 P_3 \\
& -\frac{1+x^2}{2(1+x)} \zeta_2 L_1 + \frac{2}{1+x} (L_1 P_1 - 2R_1) - 2 + 2x \Big] \quad (329)
\end{aligned}$$

$$\begin{aligned}
\tilde{E}_{11110}^{1,0} &= \frac{2}{\varepsilon^2} (L_2 - L_1) - \frac{1}{\varepsilon} \left(4\zeta_2 - \frac{7}{2} L_2^2 - 2P_1 + 2L_1 L_2 + \frac{1}{2} L_1^2 \right) - \frac{9}{2} \zeta_2 L_2 + 7\zeta_3 \\
& - \frac{1}{2} \zeta_2 L_1 - 2L_1 L_2^2 - P_1 L_2 + \frac{35}{12} L_2^3 - R_5 + 2R_1 + \frac{1}{12} L_1^3 - 6R_4 \quad (330)
\end{aligned}$$

$$\begin{aligned}
\tilde{E}_{11110}^{0,1} &= \frac{2}{\varepsilon^2} (1 + L_2) - \frac{1}{\varepsilon} \left(2 + 2\zeta_2 - 3L_2 - \frac{7}{2} L_2^2 - 2P_2 \right) - \frac{1}{2} \zeta_2 - \frac{13}{2} \zeta_2 L_2 + 3\zeta_3 \\
& + \frac{35}{12} L_2^3 + 4L_2 P_2 - 2R_2 + L_1 L_2^2 + P_1 L_2 - R_5 - P_1 - 3L_2 + \frac{7}{4} L_2^2 + 2 \quad (331)
\end{aligned}$$

$$\begin{aligned}
\tilde{E}_{11110}^{1,1} &= \frac{2x}{\varepsilon^2} (L_2 - L_1 + 1) + \frac{x}{\varepsilon} \left(2P_2 + \frac{7}{2} L_2^2 + 3L_2 + 4P_1 - 6\zeta_2 - \frac{1}{2} L_1^2 - 2 \right) \\
& - x \left[\frac{1}{2} \zeta_2 L_1 + \frac{13}{2} \zeta_2 L_2 + \frac{1}{2} \zeta_2 - 9\zeta_3 - \frac{1}{12} L_1^3 - L_1 L_2^2 - 4L_2 P_2 - \frac{35}{12} L_2^3 \right. \\
& \left. - \frac{7}{4} L_2^2 - L_2 P_1 + 3L_2 + P_1 + 2R_2 - 2R_1 + 8R_4 + R_5 - 2 \right] \quad (332)
\end{aligned}$$

$$\begin{aligned}
\tilde{E}_{11110}^{2,0} &= \frac{2}{\varepsilon^2} (xL_2 - xL_1 + 1 - x) - \frac{1}{\varepsilon} \left[2 - 2x - 4xP_1 + 6x\zeta_2 - 4(1-x)L_2 \right. \\
& \left. - 2xP_2 - (1+x)L_1 + \frac{x}{2} L_1^2 - \frac{7}{2} xL_2^2 \right] + (1-x) \left(4L_2^2 - 4L_2 - \frac{3}{2} \zeta_2 + 2 \right) \\
& + (1+x) (2P_2 - L_1 + 2L_1 L_2) - 8xR_4 + 2xR_1 - xR_5 - 2xR_2 + 9\zeta_3 x \\
& - \frac{1-3x}{4} L_1^2 + \frac{35}{12} xL_2^3 + \frac{x}{12} L_1^3 - \frac{x}{2} \zeta_2 L_1 + 3xL_2 P_2 - \frac{11}{2} x\zeta_2 L_2 \quad (333)
\end{aligned}$$

$$\begin{aligned}
\tilde{E}_{21110}^{1,1} &= \frac{2}{\varepsilon^2} (\mathcal{D}_0(x) + \delta(1-x) - 1) - \frac{2}{\varepsilon} \left[3L_2 + \frac{x}{1-x} L_1 - 3\mathcal{D}_1(x) - \mathcal{D}_0(x) \right. \\
& \left. + 1 + (1+\zeta_2) \delta(1-x) \right] + \frac{x^2}{1-x} P_2 + \left(\frac{7}{2} + 3x \right) \zeta_2 + (x-7) L_2^2 \\
& + xP_1 - 2L_2 - x \frac{4-x}{1-x} L_1 L_2 + 2 + \left(2 - \frac{3}{2} \zeta_2 + 2\zeta_3 \right) \delta(1-x) \\
& - \left(2 + \frac{7}{2} \zeta_2 \right) \mathcal{D}_0(x) + 2\mathcal{D}_1(x) + 7\mathcal{D}_2(x) \quad (334)
\end{aligned}$$

$$\begin{aligned}
\tilde{E}_{11111}^{1,0} &= \frac{1}{1-x} \left[2xP_3 - 2P_2 + 2xL_1 L_3 + \frac{x^2}{1-x} L_1^2 - (2-3x)\zeta_2 \right] \\
& + 2(\zeta_2 + \zeta_3) \delta(1-x) + \frac{(-1)^n}{1+x} \left[-2(x+2)P_3 - 2\frac{1-x}{1+x} P_2 \right. \\
& \left. - 2(x+2)L_1 L_3 - (x+2)\zeta_2 + x\frac{4+x}{1+x} L_1^2 \right] \quad (335)
\end{aligned}$$

$$\begin{aligned}
\tilde{E}_{11111}^{0,1} &= \frac{1}{1-x} \left[2(1+2x) (L_1 L_3 + P_3) - (1-3x)L_1 L_2 - (1-4x)P_1 \right] - \frac{1}{2} L_2^2 \\
& - \frac{x(5-7x)}{2(1-x)^2} L_1^2 + 2(\zeta_2 + \zeta_3) \delta(1-x) + \frac{(-1)^n}{1+x} \left[\frac{x(1-2x)}{1+x} L_1^2 + \zeta_2 x \right. \\
& \left. + 2xL_1 L_3 + \frac{2x(1-x)}{1+x} P_2 + 2xP_3 \right] \quad (336)
\end{aligned}$$

$$\begin{aligned}\tilde{E}_{11111}^{1,1} &= \frac{1}{1-x} \left[x(1+x)L_1L_2 + x(1+2x)(2P_3 + 2L_1L_3 + P_1) \right] - \frac{x}{2}L_2^2 \\ &\quad + \frac{x^2(5x-3)}{2(1-x)^2}L_1^2 - 4\zeta_2x + 2(\zeta_2 + \zeta_3)\delta(1-x) + \frac{(-1)^n}{1+x} \left[\frac{2x(1-x)}{1+x}P_2 \right. \\ &\quad \left. - \frac{x^2(4+x)}{1+x}L_1^2 + x(2+x)(2P_3 + 2L_1L_3 + \zeta_2) \right] \end{aligned} \quad (337)$$

$$\begin{aligned}\tilde{E}_{11111}^{2,0} &= \frac{1}{1-x} \left[(x^2 - 2x - 1)P_2 + 2x^2L_1L_3 + (4x^2 - 2x - 1)\zeta_2 + \frac{x(1+x^2)}{2(1-x)}L_1^2 \right. \\ &\quad \left. + 2x^2P_3 \right] + 2(\zeta_2 + \zeta_3)\delta(1-x) + \frac{(-1)^n}{1+x} \left[\frac{x(3-8x-5x^2)}{2(1+x)}L_1^2 \right. \\ &\quad \left. + (2x^2 + 2x - 1)(2P_3 + \zeta_2 + 2L_1L_3) + (x^2 + 2x - 1)\frac{1-x}{1+x}P_2 \right] \end{aligned} \quad (338)$$

The F -type integrals are given by

$$\begin{aligned}\tilde{F}_{01111}^{0,0} &= \frac{1}{(1-x)} \left[-\frac{2}{\varepsilon^2}L_2 - \frac{1}{\varepsilon} \left(\frac{1}{2}L_2^2 - 4P_2 + 4\zeta_2 \right) - \frac{5}{2}\zeta_2L_2 + 2\zeta_3 \right. \\ &\quad \left. - \frac{1}{12}L_2^3 + L_2P_2 + 2R_2 - 4R_5 \right] \end{aligned} \quad (339)$$

$$\tilde{F}_{01111}^{1,0} = x\tilde{F}_{01111}^{0,0} \quad (340)$$

$$\begin{aligned}\tilde{F}_{01111}^{0,1} &= \frac{2}{\varepsilon^2} \left(L_2 + \frac{x}{1-x} \right) - \frac{1}{\varepsilon} \left(4P_2 - \frac{1}{1-x} (4xL_1 + (2-x)L_2 - 2x) - 4\zeta_2 \right. \\ &\quad \left. - \frac{1}{2}L_2^2 \right) + \frac{5}{2}\zeta_2L_2 - 2\zeta_3 + \frac{1}{12}L_2^3 + 4R_5 - 2R_2 - L_2P_2 - \frac{4-x}{1-x}P_2 \\ &\quad + \frac{1}{1-x} \left(\frac{8-3x}{2}\zeta_2 + \frac{2-x}{4}L_2^2 + (2-3x)L_2 + x(L_1L_2 + 2L_1^2 + 2) \right) \end{aligned} \quad (341)$$

$$\tilde{F}_{01111}^{1,1} = x\tilde{F}_{01111}^{0,1} \quad (342)$$

$$\tilde{F}_{01111}^{2,0} = x^2\tilde{F}_{01111}^{0,0} \quad (343)$$

$$\begin{aligned}\tilde{F}_{01111}^{0,2} &= -\frac{1}{\varepsilon^2} \left[2(1-x)L_2 + \frac{x(2-3x)}{1-x} \right] - \frac{1}{\varepsilon} \left[(1-x) \left(\frac{1}{2}L_2^2 + 4\zeta_2 - 4P_2 \right) \right. \\ &\quad \left. + \frac{3x^2 - 10x + 6}{2(1-x)}L_2 + \frac{2x(2-3x)}{1-x}L_1 + \frac{x(3x-1)}{1-x} \right] - \frac{2x}{1-x}L_1 \\ &\quad - \frac{3x^2 - 10x + 6}{8(1-x)}L_2^2 - \frac{9x^2 - 12x + 5}{2(1-x)}L_2 - \frac{9x^2 - 38x + 24}{4(1-x)}\zeta_2 \\ &\quad - \frac{x(2-3x)}{1-x} \left(L_1^2 + \frac{1}{2}L_1L_2 \right) + \frac{3(x^2 - 6x + 4)}{2(1-x)}P_2 - \frac{x(1-6x)}{2(1-x)} \\ &\quad + (1-x) \left[2\zeta_3 - \frac{5}{2}\zeta_2L_2 - \frac{1}{12}L_2^3 + L_2P_2 + 2R_2 - 4R_5 \right] \end{aligned} \quad (344)$$

$$\tilde{F}_{10111}^{0,0} = \frac{1}{1-x} \left(\frac{4}{\varepsilon^2} - \frac{4}{\varepsilon} (1 - L_2) + 2L_2^2 - 4L_2 + \zeta_2 + 4 \right) \quad (345)$$

$$\tilde{F}_{10111}^{1,0} = x\tilde{F}_{10111}^{0,0} \quad (346)$$

$$\tilde{F}_{10111}^{0,1} = \frac{1}{1-x} \left[-\frac{2-4x}{\varepsilon^2} - \frac{1}{\varepsilon} ((2-4x)L_2 + 4x - 1) - \frac{1-2x}{2}\zeta_2 \right]$$

$$-(1-2x)L_2^2 + (1-4x)L_2 + 4x - \frac{1}{2} \Big] \quad (347)$$

$$\begin{aligned} \tilde{F}_{11011}^{1,0} = & \frac{x}{1-x} \left[-\frac{2}{\varepsilon^2}L_1 - \frac{5}{2\varepsilon}L_1^2 - \frac{1}{2}\zeta_2L_1 - 2\zeta_3 + 2L_1P_1 - \frac{7}{12}L_1^3 \right. \\ & \left. + 4L_1P_3 - 8R_3 - 4R_1 \right] \end{aligned} \quad (348)$$

$$\begin{aligned} \tilde{F}_{11011}^{0,1} = & -\frac{2}{\varepsilon^2} \left[1 + \frac{x}{1-x}L_1 \right] - \frac{1}{\varepsilon} \left[\frac{5x}{2(1-x)}L_1^2 + \frac{1-3x}{1-x}L_1 - 2 \right] \\ & + \frac{x}{1-x} \left[2L_1P_1 + 4L_1P_3 - 4R_1 - 8R_3 - \frac{1}{2}\zeta_2L_1 - 2\zeta_3 - \frac{7}{12}L_1^3 \right] \\ & - \frac{1}{2}\zeta_2 + \frac{1+9x}{4(1-x)}L_1^2 + \frac{1-3x}{1-x}L_1 - 2 \end{aligned} \quad (349)$$

$$\tilde{F}_{11011}^{1,1} = x\tilde{F}_{11011}^{0,1} \quad (350)$$

$$\tilde{F}_{11011}^{2,0} = x\tilde{F}_{11011}^{1,0} \quad (351)$$

$$\begin{aligned} \tilde{F}_{11011}^{0,2} = & \frac{1}{\varepsilon^2} \left(1 - 3x - \frac{2x^2}{1-x}L_1 \right) - \frac{1}{\varepsilon} \left(\frac{5x^2}{2(1-x)}L_1^2 - \frac{9x^2 - 4x + 1}{2(1-x)}L_1 - 3x \right) \\ & + \frac{27x^2 + 4x - 1}{8(1-x)}L_1^2 - \frac{9x^2 - 3x + 1}{2(1-x)}L_1 + \frac{x^2}{1-x} \left(4L_1P_1 + 4L_1P_3 \right. \\ & \left. + L_1^2L_2 - 6R_1 - 8R_3 - 2R_5 - \frac{1}{2}\zeta_2L_1 - \frac{7}{12}L_1^3 \right) + \frac{1-3x}{4}\zeta_2 - \frac{1}{2} - 3x \end{aligned} \quad (352)$$

$$\tilde{F}_{11101}^{0,0} = \frac{1}{1-x} \left[\frac{2}{\varepsilon^2} - \frac{1}{\varepsilon}(L_1 - 4L_2 + 2) + \frac{1}{2}\zeta_2 + 2L_2^2 + \frac{1}{4}L_1^2 + \frac{1+3x}{1-x}L_1 + 2 \right] \quad (353)$$

$$\tilde{F}_{11101}^{1,0} = x\tilde{F}_{11101}^{0,0} \quad (354)$$

$$\tilde{F}_{11101}^{0,1} = -\frac{1}{2}(1-x)\tilde{F}_{11101}^{0,0} \quad (355)$$

$$\tilde{F}_{11101}^{1,1} = x\tilde{F}_{11101}^{0,1} \quad (356)$$

$$\tilde{F}_{11101}^{2,0} = x^2\tilde{F}_{11101}^{0,0} \quad (357)$$

$$\tilde{F}_{11110}^{1,0} = \frac{x}{1-x} \left(\frac{2}{\varepsilon^2} - \frac{1}{\varepsilon}(2 - 3L_2) - \frac{1}{2}\zeta_2 + \frac{7}{4}L_2^2 - 3L_2 - P_1 + 2 \right) \quad (358)$$

$$\tilde{F}_{11110}^{1,1} = \frac{x}{2}\tilde{F}_{11110}^{1,0} - \frac{x}{2(1-x)}L_2 \quad (359)$$

$$\tilde{F}_{11110}^{2,0} = x\tilde{F}_{11110}^{1,0} \quad (360)$$

$$\tilde{F}_{11110}^{0,1} = \frac{1}{2}\tilde{F}_{11110}^{1,0} - \frac{1}{2(1-x)}L_2 \quad (361)$$

$$\tilde{F}_{21110}^{1,0} = -x(1-x)^{-2+\varepsilon} \left[-\frac{2}{\varepsilon} + 2 - \left(\frac{\zeta_2}{2} + 2 \right) \varepsilon \right] + \frac{x}{1-x} \left(\zeta_2 + P_1 + \frac{1}{2}L_2^2 \right) \quad (362)$$

$$\begin{aligned} \tilde{F}_{11111}^{1,0} = & \frac{x}{1-x} \left[-\frac{1}{2}L_2^2 + 2\frac{1+x}{1-x}(L_1L_3 + P_3) + \frac{1}{1-x} \left((3x-1)\zeta_2 - xL_1L_2 \right. \right. \\ & \left. \left. - \frac{3}{2}xL_1^2 + (1-2x)P_2 \right) \right] \end{aligned} \quad (363)$$

$$\tilde{F}_{11111}^{0,1} = \tilde{F}_{11111}^{1,0} - \frac{1}{1-x}L_2 - \frac{x}{(1-x)^2}L_1 \quad (364)$$

$$\tilde{F}_{11111}^{1,1} = x\tilde{F}_{11111}^{0,1} \quad (365)$$

$$\tilde{F}_{11111}^{2,0} = x\tilde{F}_{11111}^{1,0} \quad (366)$$

References

- [1] F. A. Berends, W. L. van Neerven and G. J. H. Burgers, Nucl. Phys. B **297** (1988) 429 [Erratum-ibid. B **304** (1988) 921].
- [2] A. Arbuzov, D. Y. Bardin, J. Blümlein, L. Kalinovskaya and T. Riemann, Comput. Phys. Commun. **94** (1996) 128 [arXiv:hep-ph/9511434];
J. Blümlein and H. Kawamura, Phys. Lett. B **553** (2003) 242 [arXiv:hep-ph/0211191].
- [3] M. Preger and F. Murtas, ADONE (1969–1993),
<http://www.lnf.infn.it/acceleratori/adone/>
- [4] S. Williams, CERN Courier, June 1st (2003),
<http://cerncourier.com/cws/article/cern/28865>;
<http://www-ssrl.slac.stanford.edu/spear3/>
- [5] S. L. Wu, Phys. Rept. **107** (1984) 59;
B. Adeva *et al.* [Mark-J Collaboration], Phys. Rept. **109** (1984) 131;
B. Naroska, Phys. Rept. **148** (1987) 67.
- [6] Physics at LEP, eds. J. Ellis and R. Peccei, CERN Report 86–02;
ECFA Workshop on LEP 200, CERN 87–08, Z Physics at LEP1, eds. G. Altarelli, R. Kleiss, and C. Verzegnassi, CERN Report 89-08;
Physics at LEP200 and Beyond, Nucl. Phys. **37B** (1994), eds. T. Riemann and J. Blümlein;
Reports of the Working Group on Precision Calculations for the Z-Resonance, eds. D. Bardin, W. Hollik, and G. Passarino, CERN Report 95–03;
Physics at LEP2, eds. G. Altarelli, T. Sjöstrand, and F. Zwirner, CERN Report 96-01;
O. Adriani *et al.* [L3 Collaboration], Phys. Rept. **236** (1993) 1;
S. Schael *et al.*, ALEPH, DELPHI, L3, OPAL Collaborations *et al.*, Phys. Rept. **427** (2006) 257 [arXiv:hep-ex/0509008].
- [7] P. C. Rowson, D. Su and S. Willocq, Ann. Rev. Nucl. Part. Sci. **51** (2001) 345 [arXiv:hep-ph/0110168].
- [8] J. A. Aguilar-Saavedra *et al.* [ECFA/DESY LC Physics Working Group], arXiv:hep-ph/0106315;
International Linear Collider Reference Design Report, ILC-REPORT-2007-001, Eds. J. Brau, Y. Okada, and N. Walker; Vol. 1–4.
G. Aarons *et al.* [ILC Collaboration], International Linear Collider Reference Design Report Volume 2: Physics At The ILC, [arXiv:0709.1893 [hep-ph]].
- [9] S. van der Meer, The CLIC Project and the Design for an e^+e^- Collider, CLIC-NOTE-68, (1988);
R.W. Assmann *et al.*, CLIC Study Team, A 3 TeV e^+e^- Linear Collider Based on CLIC Technology, CERN 2000-008;
E. Accomando *et al.* [CLIC Physics Working Group], arXiv:hep-ph/0412251.
- [10] E. Accomando *et al.* [ECFA/DESY LC Physics Working Group], Phys. Rept. **299** (1998) 1 [arXiv:hep-ph/9705442].
- [11] P. Franzini and M. Moulson, Ann. Rev. Nucl. Part. Sci. **56** (2006) 207 [arXiv:hep-ex/0606033].

- [12] For other e^+e^- accelerators see e.g.
http://en.wikipedia.org/wiki/List_of_accelerators_in_particle_physics.
- [13] see e.g. : A. Akhundov, D. Bardin, and T. Riemann, Nucl. Phys. **B276** (1986) 1;
A. Barroso et al. Electroweak radiative corrections at LEP energies, CERN-EP-87-70, 1987
and references therein;
F. Berends, G. Burgers, W. Hollik, and W.L. van Neerven, Phys. Lett. **B203** (1988) 177;
D. Bardin, A. Leike, T. Riemann, and M. Sachwitz, Phys. Lett. **B206** (1988) 539;
M. Greco, Riv. Nuovo Cim. **11** N5(1988) 1;
M. Consoli, W. Hollik, and F. Jegerlehner, Electroweak Radiative Corrections for Z Physics
CERN-TH-5527-89, LEP Physics Workshop (1989) and references therein;
W. Hollik, Fortsch. Phys. **38** (1990) 165;
S. Jadach and B. Ward, Comput. Phys. Commun. **56** (1990) 351, **66** (1991) 276; **79** (1994)
503; **124** (2000) 233;
S. Jadach, E. Richter-Was, B. Ward, and Z. Was, Comput. Phys. Commun. **70** (1992) 305;
S. Jadach, W. Plazcek, E. Richter-Was, B. Ward, and Z. Was, Comput. Phys. Commun.
102 (1997) 229;
S. Jadach, W. Plazcek, M. Skrzypek, B. Ward, and Z. Was, Comput. Phys. Commun. **140**
(2001) 432;
D. Bardin et al. Phys. Lett. **B255** (1991) 290;
D. Bardin and G. Passarino, *The Standard Model in the Making*, (Oxford, Calendron, 1999);
D. Bardin et al., hep-ph/9412201; Comput. Phys. Commun. **133** (2001) 229.
- [14] W. Beenakker, F. A. Berends, W. L. van Neerven, *Applications Of Renormalization Group
Methods To Radiative Corrections*, in: *Proceedings of the Ringberg Workshop, 1989*,
(Springer, Berlin, 1989), pp.3, Ed. J.H. Kühn;
M. Skrzypek, Acta Phys. Polon. **B23** (1992) 135;
M. Jezabek, Z. Phys. **C56** (1992) 285;
M. Przybycien, Acta Phys. Polon. **B24** (1993) 1105, [arXiv:hep-th/9511029];
J. Blümlein, Z. Phys. C **65** (1995) 293 [arXiv:hep-ph/9403342];
J. Blümlein, S. Riemersma and A. Vogt, Eur. Phys. J. C **1** (1998) 255
[arXiv:hep-ph/9611214];
A. B. Arbuzov, Phys. Lett. **B470** (1999) 252 [arXiv:hep-ph/9908361].
- [15] J. Blümlein and H. Kawamura, Eur. Phys. J. C **51** (2007) 317 [arXiv:hep-ph/0701019];
Nucl. Phys. B **708** (2005) 467 [arXiv:hep-ph/0409289].
- [16] K. Melnikov and A. Mitov, Phys. Rev. D **70** (2004) 034027 [arXiv:hep-ph/0404143].
- [17] A. Arbuzov, K. Melnikov, Phys. Rev. **D66** (2002) 093003. [hep-ph/0205172];
C. Anastasiou, K. Melnikov, F. Petriello, JHEP **0709** (2007) 014. [hep-ph/0505069].
- [18] M. Buza, Y. Matiounine, J. Smith, R. Migneron and W. L. van Neerven, Nucl. Phys. B **472**
(1996) 611 [arXiv:hep-ph/9601302];
- [19] I. Bierenbaum, J. Blümlein and S. Klein, Nucl. Phys. B **780** (2007) 40
[arXiv:hep-ph/0703285].
- [20] R. Hamberg, W. L. van Neerven and T. Matsuura, Nucl. Phys. B **359** (1991) 343 [Erratum-
ibid. B **644** (2002) 403].

- [21] R. V. Harlander and W. B. Kilgore, Phys. Rev. Lett. **88** (2002) 201801 [arXiv:hep-ph/0201206].
- [22] K.G. Wilson, Phys. Rev. **179** (1969) 1699;
R.A. Brandt and G. Preparata, Fortschr. Phys. **18** (1970) 249;
W. Zimmermann, Lect. on Elementary Particle Physics and Quantum Field Theory, Brandeis Summer Inst., Vol. 1, (MIT Press, Cambridge, 1970), p. 395;
Y. Frishman, Ann. Phys. **66** (1971) 373.
- [23] E. Laenen, S. Riemersma, J. Smith and W. L. van Neerven, Nucl. Phys. B **392** (1993) 229.
- [24] S. I. Alekhin and J. Blümlein, Phys. Lett. B **594** (2004) 299 [arXiv:hep-ph/0404034].
- [25] B. A. Kniehl, M. Krawczyk, J. H. Kuhn, R. G. Stuart, Phys. Lett. **B209** (1988) 337.
- [26] M. Buza, Y. Matiounine, J. Smith and W. L. van Neerven, Eur. Phys. J. C **1** (1998) 301, [hep-ph/9612398]; Nucl. Phys. **B485** (1997) 420, [hep-ph/9608342];
I. Bierenbaum, J. Blümlein and S. Klein, Nucl. Phys. B **780** (2007) 40, [hep-ph/0703285];
Phys. Lett. B **672** (2009) 401, [hep-ph/0901.0669]; arXiv:1008.0792 [hep-ph];
I. Bierenbaum, J. Blümlein, S. Klein and C. Schneider, Nucl. Phys. B **803** (2008) 1, [hep-ph/0803.0273];
J. Blümlein, A. De Freitas, W. L. van Neerven and S. Klein, Nucl. Phys. B **755** (2006) 272, [hep-ph/0608024].
- [27] I. Bierenbaum, J. Blümlein and S. Klein, Nucl. Phys. B **820** (2009) 417, [hep-ph/0904.3563];
S. Klein, Mellin moments of heavy flavor contributions to $F_2(x, Q^2)$ at NNLO, PhD Thesis, TU Dortmund, September 2009, [arXiv:0910.3101 [hep-ph]].
- [28] I. Bierenbaum, J. Blümlein and S. Klein, arXiv:1008.0792 [hep-ph];
J. Blümlein, S. Klein and B. Tödtli, Phys. Rev. D **80** (2009) 094010, [arXiv:0909.1547 [hep-ph]];
J. Ablinger, I. Bierenbaum, J. Blümlein, A. Hasselhuhn, S. Klein, C. Schneider, and F. Wißbrock, Nucl. Phys. (Proc. Suppl.) (2010) 242, [arXiv:1007.0375 [hep-ph]];
J. Ablinger, J. Blümlein, S. Klein, C. Schneider and F. Wißbrock, Nucl. Phys. B **844** (2011) 26 [arXiv:1008.3347 [hep-ph]].
- [29] M. Böhm, A. Denner, and H. Joos, Gauge Theories of the Strong and Electroweak Interaction, (B.G. Teubner, Stuttgart, 2001).
- [30] B. Geyer, D. Robaschik, E. Wieczorek, Fortsch. Phys. **27** (1979) 75.
- [31] J. Blümlein, V. Ravindran, Nucl. Phys. **B716** (2005) 128 [hep-ph/0501178].
- [32] J. Blümlein and S. Kurth, Phys. Rev. D **60** (1999) 014018, [arXiv:hep-ph/9810241].
- [33] J. A. M. Vermaseren, Int. J. Mod. Phys. A **14** (1999) 2037, [arXiv:hep-ph/9806280].
- [34] J. Blümlein, D. J. Broadhurst and J. A. M. Vermaseren, Comput. Phys. Commun. **181** (2010) 582 [arXiv:0907.2557 [math-ph]] and references therein.
- [35] J. Blümlein, Comput. Phys. Commun. **159** (2004) 19 [arXiv:hep-ph/0311046]; Comput. Phys. Commun. **180** (2009) 2218 [arXiv:0901.3106 [hep-ph]]; Structural Relations of Harmonic Sums and Mellin Transforms at Weight $w=6$, Proceedings of the Conference on Motives,

Quantum Field Theory, and Pseudodifferential Operators, held at the Clay Mathematics Institute, Boston University, June 2–13, 2008, Clay Mathematics Proceedings **12** (2010) 167, Eds. A. Carey, D. Ellwood, S. Paycha, S. Rosenberg, [arXiv:0901.0837 [math-ph]].

- [36] J. Blümlein, Comput. Phys. Commun. **133** (2000) 76 [arXiv:hep-ph/0003100];
J. Blümlein and S. O. Moch, Phys. Lett. B **614** (2005) 53 [arXiv:hep-ph/0503188];
A. V. Kotikov, V. N. Velizhanin, [hep-ph/0501274];
S. Albino, Phys. Lett. **B674** (2009) 41-48. [arXiv:0902.2148 [hep-ph]].
- [37] C. G. Callan, Phys. Rev. **D2** (1970) 1541;
K. Symanzik, Commun. Math. Phys. **18** (1970) 227.
- [38] H. D. Politzer, Phys. Rev. Lett. **30** (1973) 1346;
D. J. Gross and F. Wilczek, Phys. Rev. Lett. **30** (1973) 1343;
W. E. Caswell, Phys. Rev. Lett. **33** (1974) 244;
D. R. T. Jones, Nucl. Phys. B **75** (1974) 531.
- [39] E. Fermi, Z. Phys. **29** (1924) 315;
E. J. Williams, Phys. Rev. **45** (1934) 729;
C. F. von Weizsäcker, Z. Phys. **88** (1934) 61;
D. J. Gross and F. Wilczek, Phys. Rev. D **8** (1973) 3633; D **9** (1974) 980;
H. Georgi and H. D. Politzer, Phys. Rev. D **9** (1974) 416.
- [40] E. G. Floratos, D. A. Ross and C. T. Sachrajda, Nucl. Phys. B **129** (1977) 66 [Erratum-ibid. B **139** (1978) 545]; Nucl. Phys. B **152** (1979) 493;
A. Gonzalez-Arroyo, C. Lopez and F. J. Yndurain, Nucl. Phys. B **153** (1979) 161;
A. Gonzalez-Arroyo and C. Lopez, Nucl. Phys. B **166** (1980) 429;
E. G. Floratos, C. Kounnas and R. Lacaze, Nucl. Phys. B **192** (1981) 417;
G. Curci, W. Furmanski and R. Petronzio, Nucl. Phys. B **175** (1980) 27;
W. Furmanski and R. Petronzio, Phys. Lett. B **97** (1980) 437;
R. Hamberg and W. L. van Neerven, Nucl. Phys. B **379** (1992) 143;
R. K. Ellis and W. Vogelsang, arXiv:hep-ph/9602356;
S. Moch, J. A. M. Vermaseren, Nucl. Phys. **B573** (2000) 853-907. [hep-ph/9912355].
- [41] N. Gray, D. J. Broadhurst, W. Gräfe and K. Schilcher, Z. Phys. C **48** (1990) 673;
D. J. Broadhurst, N. Gray and K. Schilcher, Z. Phys. C **52** (1991) 111;
K. Melnikov and T. van Ritbergen, Nucl. Phys. B **591** (2000) 515 [arXiv:hep-ph/0005131];
P. Marquard, L. Mihaila, J. H. Piclum and M. Steinhauser, Nucl. Phys. B **773** (2007) 1 [arXiv:hep-ph/0702185].
- [42] L. F. Abbott, Nucl. Phys. B **185** (1981) 189;
A. Rebhan, Z. Phys. C **30** (1986) 309;
F. Jegerlehner and O. V. Tarasov, Nucl. Phys. B **549** (1999) 481 [arXiv:hep-ph/9809485].
- [43] K. Yoshida, Functional Analysis, (Springer, Berlin, 1978), Chapt. XI;
L. Schwartz, Théorie des Distributions, Vol. I,II, (Hermann & Cie, Paris, 1951);
V.S. Vladimirov, Gleichungen der Mathematischen Physik, (DVW, Berlin, 1972); (Nauka, Moscow, 1967).
- [44] B. Humpert and W. van Neerven, Nucl. Phys. **B184** (1981) 225.

- [45] M.E. Peskin and D.V. Schroeder, *An Introduction to Quantum Field Theory*, (Addison Wesley, New York, 1995).
- [46] J. A. M. Vermaseren, arXiv:math-ph/0010025.
- [47] R. Hamberg, *Second order gluonic contributions to physical quantities*. PhD Thesis.
- [48] J. Lagrange *Nouvelles recherches sur la nature et la propagation du son*, Miscellanea Taurinensis, t. II, 1760-61; Oeuvres t. I, p. 263;
 C.F. Gauss, *Theoria attractionis corporum sphaeroidicorum ellipticorum homogeneorum methodo novo tractata*, Commentationes societas scientiarum Gottingensis recentiores, Vol III, 1813, Werke Bd. V pp. 5-7;
 G. Green, *Essay on the Mathematical Theory of Electricity and Magnetism*, Nottingham, 1828 [Green Papers, pp. 1-115];
 M. Ostrogradski, *Mem. Ac. Sci. St. Peters.*, **6**, (1831) 39;
 G. 't Hooft, M. J. G. Veltman, *Nucl. Phys.* **B44** (1972) 189;
 K. G. Chetyrkin, A. L. Kataev, F. V. Tkachov, *Nucl. Phys.* **B174** (1980) 345;
 S. Laporta, *Int. J. Mod. Phys.* **A15** (2000) 5087-5159. [hep-ph/0102033].
- [49] R. Mertig and R. Scharf, *Comput. Phys. Commun.* **111** (1998) 265 [arXiv:hep-ph/9801383] and updates.
- [50] N. Nielsen, *Nova Acta Leopoldiana (Halle)* 90 (1909) 123;
 K.S. Kölbig, J.A. Mignaco, E. Remiddi, *BIT* 10 (1970) 38;
 K.S. Kölbig, *SIAM J. Math. Anal.* **17** (1986) 1232;
 L. Lewin, *Polylogarithms and Associated Functions*, (North Holland, Amsterdam, 1981).
- [51] H. D. Politzer, *Phys. Rept.* **14** (1974) 129;
 J. Blümlein and N. Kochelev, *Nucl. Phys. B* **498** (1997) 285 [arXiv:hep-ph/9612318].
- [52] J. Blümlein, V. Ravindran and W. L. van Neerven, *Nucl. Phys. B* **586** (2000) 349 [arXiv:hep-ph/0004172].
- [53] W. L. van Neerven, A. Vogt, *Nucl. Phys.* **B603** (2001) 42, [hep-ph/0103123].
- [54] R.B. Paris and D. Kaminski, *Asymptotics and Mellin-Barnes Integrals*, (Cambridge University Press, Cambridge, 2001).
 V. A. Smirnov, *Feynman integral calculus*, (Springer, Berlin, 2006);
 V. A. Smirnov, *Phys. Lett. B* **460** (1999) 397 [arXiv:hep-ph/9905323];
 V. A. Smirnov and O. L. Veretin, *Nucl. Phys. B* **566** (2000) 469 [arXiv:hep-ph/9907385];
 J. B. Tausk, *Phys. Lett. B* **469** (1999) 225 [arXiv:hep-ph/9909506].
- [55] I. Bierenbaum and S. Weinzierl, *Eur. Phys. J. C* **32** (2003) 67.
- [56] I. Bierenbaum, J. Blümlein and S. Klein, *Phys. Lett. B* **648** (2007) 195, [hep-ph/0702265];
- [57] M. Czakon, *Comput. Phys. Commun.* **175** (2006) 559.
- [58] W. Bailey *Generalized Hypergeometric Series*, (Cambridge University Press, Cambridge, 1935), 108 p.;
 L. Slater *Generalized Hypergeometric Functions*, (Cambridge University Press, Cambridge, 1966), 273 p.

- [59] I. M. Ryshik and I. S. Gradstein, *Tables of Series, Products, and Integrals*, (DVW, Berlin, 1957).
- [60] A. Devoto and D. W. Duke, *Riv. Nuovo Cim.* **7N6** (1984) 1.